

# Operator product expansion algebra

Jan Holland\* and Stefan Hollands†

*School of Mathematics, Cardiff University  
Cardiff, United Kingdom*

March 25, 2013

## Abstract

We establish conceptually important properties of the operator product expansion (OPE) in the context of perturbative, Euclidean  $\varphi^4$ -quantum field theory. First, we demonstrate, generalizing earlier results and techniques of hep-th/1105.3375, that the 3-point OPE,  $\langle \mathcal{O}_{A_1} \mathcal{O}_{A_2} \mathcal{O}_{A_3} \rangle = \sum_C \mathcal{C}_{A_1 A_2 A_3}^C \langle \mathcal{O}_C \rangle$ , usually interpreted only as an asymptotic short distance expansion, actually *converges* at *finite*, and even large, distances. We further show that the factorization identity  $\mathcal{C}_{A_1 A_2 A_3}^B = \sum_C \mathcal{C}_{A_1 A_2}^C \mathcal{C}_{C A_3}^B$  is satisfied for suitable configurations of the spacetime arguments. Again, the infinite sum is shown to be convergent. Our proofs rely on explicit bounds on the remainders of these expansions, obtained using refined versions, mostly due to Kopper et al., of the renormalization group flow equation method. These bounds also establish that each OPE coefficient is a real analytic function in the spacetime arguments for non-coinciding points. Our results hold for arbitrary but finite loop orders. They lend support to proposals for a general axiomatic framework of quantum field theory, based on such ‘consistency conditions’ and akin to vertex operator algebras, wherein the OPE is promoted to the defining structure of the theory.

## 1 Introduction

The operator product expansion (OPE) [1, 2, 3]

$$\mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) \sim \sum_C \mathcal{C}_{A_1 \dots A_N}^C(x_1, \dots, x_N) \mathcal{O}_C(x_N), \quad (1.1)$$

---

\*HollandJW1@Cardiff.ac.uk

†HollandsS@Cardiff.ac.uk

is an important structure in quantum field theory, both for practical calculations, as well as also from a conceptual point of view. It is normally understood as a statement about operator insertions into any, suitably well-behaved, quantum state (in the Lorentzian context), or as an insertion into the vacuum functional with any number of additional “spectator fields” (in the Euclidean context). The symbols  $\mathcal{O}_A$  denote the composite fields that appear in the given theory, where the label  $A$  also incorporates the tensor or spinor character of the field. In the Euclidean context, which we will focus on in this paper, the points  $x_i \in \mathbb{R}^4$  are mutually distinct.

In the original papers, it was suggested that the OPE should be an asymptotic expansion, after insertion into the vacuum functional with spectator fields. Namely, the remainder in such an expansion, truncated at a sufficiently high dimension  $D$  for the operators  $\mathcal{O}_C$  on the right side, should go to zero as  $x_1, \dots, x_{N-1} \rightarrow x_N$ , at a rate that improves as we increase  $D$ . However, it has recently been shown [4] that, at least to any fixed loop order in perturbation theory, and  $N = 2$ , the expansion is even *convergent* for *finite* separation of  $x_1, x_2$ , i.e. we can put “=” instead of “ $\sim$ ” in (1.1). This result is complemented by results of [5] obtained in an axiomatic setting of quantum field theory (QFT), as well as also by results in the context of vertex operator algebras [6] describing  $2d$  conformal field theories.

The *first result* of this paper is to generalize the convergence statement in perturbative QFT to more than just  $N = 2$  fields in the product (1.1). As in [4], we will restrict attention for simplicity and concreteness to the case of  $g\varphi^4$ -theory in  $4d$  with mass<sup>1</sup>  $m > 0$ , and to keep the discussion reasonably simple, we will also restrict to  $N = 3$  points, where the key technical differences in the proof are already visible. The composite fields in the theory are

$$\mathcal{O}_A = \partial^{w_1} \varphi \cdots \partial^{w_n} \varphi, \quad A = \{n, w\} \quad (1.2)$$

with  $w$  a multiindex, see our Notations and Conventions section for more on multi-indices. The engineering dimension of such a field is defined as usual by

$$[A] = n + \sum_i |w_i|. \quad (1.3)$$

The result is:

**Theorem 1.** *Let  $f_p(x)$  be any smooth function on  $x \in \mathbb{R}^4$  such that the support of the Fourier transform  $\hat{f}_p(q)$  is contained in a ball  $|p - q| \leq 1$  around  $p \in \mathbb{R}^4$ . Define the smeared spectator fields by*

$$\varphi(f_p) \equiv \int d^4x \, \varphi(x) f_p(x) .$$

*Then the remainder of the OPE, carried out up to operators of dimension  $D = \sum_{i=1}^3 [A_i] + \Delta$ ,*

---

<sup>1</sup>This restriction is made to simplify our proofs; we believe that it can be removed using methods along the lines of [7].

at  $l$ -loops, is bounded by

$$\begin{aligned}
& \left| \left\langle \mathcal{O}_{A_1}(x_1) \mathcal{O}_{A_2}(x_2) \mathcal{O}_{A_3}(x_3) \varphi(f_{p_1}) \cdots \varphi(f_{p_n}) \right\rangle \right. \\
& \quad \left. - \sum_{[C] \leq D} \mathcal{C}_{A_1 A_2 A_3}^C(x_1, x_2, x_3) \left\langle \mathcal{O}_C(x_3) \varphi(f_{p_1}) \cdots \varphi(f_{p_n}) \right\rangle \right| \\
& \leq m^{n-1} \prod_{i=1}^3 [A_i]! \prod_j \sup |\hat{f}_{p_j}| \sup(1, \frac{|\vec{p}|_n}{m})^{(4 \sum [A_i] + 2\Delta)(n+l+9/2)+3n} \\
& \quad \times \sum_{\lambda=0}^{2l+n/2+1} \frac{\log^\lambda \sup(1, \frac{|\vec{p}|_n}{m})}{2^\lambda \lambda!} \\
& \quad \times \frac{1}{\sqrt{\Delta}!} \frac{(\tilde{K} m \max(|x_1 - x_2|, |x_2 - x_3|, |x_1 - x_3|))^{\sum [A_i] + 1 + \Delta}}{(\min(|x_1 - x_2|, |x_2 - x_3|, |x_1 - x_3|))^{\sum [A_i] + 1}}
\end{aligned} \tag{1.4}$$

in  $g\varphi^4$ -theory. Here  $[A]$  denotes the canonical dimension of a composite field  $\mathcal{O}_A$  as in eq. (1.3).  $\tilde{K}$  is a constant depending on  $n$  and  $l$ .

From the bound one can draw the conclusion that the OPE, i.e. the sum over  $C$  in (1.4), converges for arbitrary finite distances  $|x_i - x_j| > 0$  for  $i \neq j$ , including large distances (!), because evidently  $(\text{any number})^\Delta / \sqrt{\Delta}! \rightarrow 0$ . In this regard, our result is a generalization of our earlier result [4] for  $N = 2$  points. Another important conclusion, which has no analog for  $N = 2$  points, is the following. Suppose we expand the OPE to operators up to a fixed dimension  $D = [A_1] + [A_2] + [A_3] + \Delta$ . Suppose that  $|x_i - x_j| = O(\epsilon)$  for  $i \neq j$ , and let  $\epsilon \rightarrow 0$ . Then the remainder goes as  $O(\epsilon^\Delta)$ , i.e. goes to zero e.g. for  $\Delta = 1$ . However, suppose now instead that  $|x_2 - x_3| = |x_1 - x_3| = O(\epsilon)$ , but  $|x_1 - x_2| = O(\epsilon^2)$ , i.e. one pair of points is much closer together than the remaining ones. Then, as one can see from the structure of the last line in equation (1.4), the bound will *not* go to zero as  $\epsilon \rightarrow 0$ , unless we make  $\Delta$  considerably bigger. In that case, convergence of the remainder is not guaranteed by our bound. Thus, we see that the sense in which the OPE approximates a correlator for short distances may depend on exactly how the relative distances between the points are taken to zero, when more than two insertions are present in the correlator.

The *second result* of this paper clarifies the nature of the algebraic relations that are satisfied by the OPE coefficients  $\mathcal{C}_{A_1 \dots A_N}^C$ , again in the context of perturbation theory. An obvious general expectation is that, since these coefficients resemble the structure constants of an algebra, there should hold certain associativity conditions. For example, consider a product of three fields  $\mathcal{O}_{A_1}(x_1) \mathcal{O}_{A_2}(x_2) \mathcal{O}_{A_3}(x_3)$ , which we may expand as in eq. (1.1). Now, suppose that  $|x_1 - x_2|$  is smaller than  $|x_2 - x_3|$ . Then it seems natural to first expand the product  $\mathcal{O}_{A_1}(x_1) \mathcal{O}_{A_2}(x_2)$ , regarding  $\mathcal{O}_{A_3}(x_3)$  as merely a spectator, and then to expand the result of this OPE times the spectator in a subsequent OPE. One would expect these expansions to agree, and our theorem shows that this expectation is correct:

**Theorem 2.** *Up to any arbitrary but fixed loop order  $l$  in  $g\varphi^4$ -theory, the identity*

$$\mathcal{C}_{A_1 A_2 A_3}^B(x_1, x_2, x_3) = \sum_C \mathcal{C}_{A_1 A_2}^C(x_1, x_2) \mathcal{C}_{C A_3}^B(x_2, x_3) \quad (1.5)$$

*holds for all configurations satisfying*

$$0 < \frac{|x_1 - x_2|}{|x_2 - x_3|} < \frac{1}{\tilde{K}}$$

*for some (sufficiently large) constant  $\tilde{K} > 0$  (depending on  $l, B$ ).*

In particular, implicit in this statement is the claim that the infinite sum over  $C$  on the right side of formula (1.5) converges. In the free field theory, our result is fairly trivial and we may in fact take  $\tilde{K} = 1$ . However, in the presence of interaction the result appears to us rather non-trivial and we can only show that convergence occurs if  $\tilde{K}$  is a rather large numerical constant, i.e. it appears that  $|x_1 - x_2|$  must be *much* smaller than  $|x_2 - x_3|$ .

As in the previous paper [4], our proof is based on the use of the Wilson-Wegner-Polchinski renormalization group flow equation method [8, 9, 1, 2, 10] and the powerful refinements of this method due to Kopper et al., see e.g. [11] for a review. In this method, one first introduces an infrared cutoff called  $\Lambda$ , and an ultraviolet cutoff called  $\Lambda_0$ . One then defines the quantities of interest such as correlation functions, OPE coefficients etc. for finite values of the cutoffs, and derives for them a flow equation as a function of  $\Lambda$ . This equation is then integrated subject to boundary conditions which play the role of the renormalization conditions. The quantities of interest may be bounded inductively and uniformly in the ultraviolet cutoff  $\Lambda_0$ . The last fact makes it possible to remove the cutoff and at the same time provides non-trivial bounds for them. These bounds are shown to imply the above two theorems. A side result that might be of independent interest is that the Schwinger functions with up to three insertions, as well as the OPE coefficients of up to three composite operators, are analytic functions of  $x_i$  away from coinciding configurations (where they are of course singular). These results are discussed briefly below the corresponding bounds, see corollaries 1 and 2.

It is evident that (1.5) is only one in an entire hierarchy of more general consistency conditions: We can consider more than  $N = 3$  points, and we can consider, out of  $N$  points, a proper subset of points  $(x_{i_1}, \dots, x_{i_r})$  where  $\{i_1, \dots, i_r\} \subset \{1, \dots, N\}$ , such that the distances  $|x_{i_k} - x_{i_l}|$  within the group are much smaller than the distance to any point outside the group. The, fairly obvious, consistency conditions arising in this way have been written out e.g. in [12], and are reviewed for completeness in the next section of this paper. The methods developed in this paper are capable also of proving the validity of such generalized consistency conditions in perturbative QFT. However, it seemed to us that the proof for three points was already illustrative of all the essential ingredients of our methods, while being more easy to follow. We therefore will not consider the general case in this paper.

We also strongly believe that our results are still valid if Euclidean space  $\mathbb{R}^4$  is replaced by a general real analytic Riemannian 4-manifold  $(M, g)$ , with the distance between points defined in terms of the metric in the usual way.

The paper is organized as follows: To put our results into context, we first recall the general nature of the consistency conditions that one expects to hold in QFT. Then we briefly introduce our notation and the general framework based on the renormalization group flow equations, which will be used throughout this paper. In section 4 we will then put this framework to use and derive bounds on various quantities of interest. With these estimates at hand, we are able to derive our two main results, theorems 1 and 2, in section 5 and 6, respectively.

**Notations and conventions:** Our convention for the Fourier transform in  $\mathbb{R}^4$  is

$$f(x) = \int_p \hat{f}(p) e^{ipx} := \int_{\mathbb{R}^4} \frac{d^4 p}{(2\pi)^4} e^{ipx} \hat{f}(p) . \quad (1.6)$$

We also use a standard multi-index notation. Our multi-indices are elements  $w = (w_1, \dots, w_n) \in \mathbb{N}^{4n}$ , so that each  $w_i \in \mathbb{N}^4$  is a four-dimensional multiindex whose entries are  $w_{i,\mu} \in \mathbb{N}$  and  $\mu = 1, \dots, 4$ . If  $f(\vec{p})$  is a smooth function on  $\mathbb{R}^{4n}$ , we set

$$\partial^w f(\vec{p}) = \prod_{i,\mu} \left( \frac{\partial}{\partial p_{i,\mu}} \right)^{w_{i,\mu}} f(\vec{p}) \quad (1.7)$$

and

$$w! = \prod_{i,\mu} w_{i,\mu}!, \quad |w| = \sum_{i,\mu} w_{i,\mu} . \quad (1.8)$$

We often need to take derivatives  $\partial^w$  of a product of functions  $f_1 \dots f_r$ . Using the Leibniz rule, such derivatives get distributed over the factors resulting in the sum of all terms of the form  $c_{\{v_i\}} \partial^{v_1} f_1 \dots \partial^{v_r} f_r$ , where each  $v_i$  is now a  $4n$ -dimensional multi-index, where  $v_1 + \dots + v_r = w$ , and where

$$c_{\{v_i\}} = \frac{(v_1 + \dots + v_r)!}{v_1! \dots v_r!} \leq r^{|w|} \quad (1.9)$$

is the associated weight factor. We will denote sets of indices by  $I = \{i_1, \dots, i_k\}$  with  $i_j \in \mathbb{N}$  and we denote their cardinality by  $|I|$ .

For a given set of momenta  $(p_1, \dots, p_n) \in \mathbb{R}^{4n}$  we use the shorthand notation

$$\vec{p} := (p_1, \dots, p_n) \quad , \quad |\vec{p}|_n := \sup_{J \subseteq \{1, \dots, n\}} \left| \sum_{i \in J} p_i \right| \quad , \quad \vec{p}_{n+2} := (\vec{p}, k, -k) \quad (1.10)$$

Later we will often simply write  $|\vec{p}|$  instead of  $|\vec{p}|_n$ . Further we define  $\kappa := \sup(\Lambda, m)$  for later convenience. We also often use the notation  $\log_+(x) = \log(\sup(1, x))$ .

If  $F(\varphi)$  is a differentiable function (in the Frechet space sense) of the Schwartz space function  $\varphi \in \mathcal{S}(\mathbb{R}^4)$ , we denote its functional derivative as

$$\frac{d}{dt}F(\varphi + t\psi)|_{t=0} = \int d^4x \frac{\delta F(\varphi)}{\delta \varphi(x)} \psi(x) , \quad \psi \in \mathcal{S}(\mathbb{R}^4) , \quad (1.11)$$

where the right side is understood in the sense of distributions in  $\mathcal{S}'(\mathbb{R}^4)$ . Multiple functional derivatives are denoted in a similar way and define in general distributions on multiple Cartesian copies of  $\mathbb{R}^4$ .

## 2 Consistency conditions

Before we start with the proofs of our two theorems, we would like to pause for a moment to indicate in somewhat more detail the status of these results within the general framework of quantum field theory (QFT). One perspective on QFT is to emphasize its algebraic structure. The oldest mathematical framework of this viewpoint is Algebraic Quantum Field Theory [13], wherein the observables are members of a net of  $C^*$ -algebras, whereas the states (‘Hilbert space’) are vectors in the various representations of such a net. Another manifestation of this idea, developed for  $2d$  conformal field theories, are “Vertex Operator Algebras”, see e.g. [6], and their representation theory. A generalization of this type of algebraic framework, wherein the OPE is raised to the status of the defining structure of the theory, but which is applicable in principle more generally to non-conformally invariant Euclidean QFT’s in  $d$  dimensions, was proposed e.g. in [12].

In this framework, one considers the collection of composite fields  $\mathcal{O}_A$  within a given theory as being in one-to-one correspondence with basis elements of an abstract vector space,  $V$ , which is graded by the dimensions of the fields

$$V = \bigoplus_{\Delta} V_{\Delta} , \quad (2.12)$$

with  $\Delta \in \mathbb{R}_+$  from the (countable) set of possible dimensions of the fields, and<sup>2</sup>  $\dim V_{\Delta} < \infty$ . The OPE-coefficients  $\mathcal{C}_{A_1 \dots A_N}^B(x_1, \dots, x_N)$  may then be viewed, for each collection of distinct points  $x_i \in \mathbb{R}^d$ , as the components of a linear map

$$\mathcal{C}(x_1, \dots, x_N) : V^{\otimes N} \rightarrow V , \quad (2.13)$$

which depends (real) analytically on  $(x_1, \dots, x_N)$  within the set  $C_N = \{(x_1, \dots, x_N) \mid x_i \neq x_j\}$ . Natural conditions on this collection (all  $N$ ) of linear maps were proposed in [12]. In particular, the hierarchy of  $\mathcal{C}$ ’s should reflect the algebraic nature of QFT. This is formulated

---

<sup>2</sup>One might further wish to complement this condition by a more stringent finiteness condition of the type  $\sum_{\Delta} \dim V_{\Delta} q^{\Delta} < \infty$  for  $0 < q < 1$ .

by imposing consistency conditions, of the kind described in the introduction, on the  $\mathcal{C}$ 's. For this, one considers a partition  $\mathcal{I} = (I_j)_{j=1}^n$ ,  $I_j \subset \{1, \dots, N\}$  of the set  $\{1, \dots, N\}$  into disjoint subsets<sup>3</sup> and a subset of  $C_{\mathcal{I}}$  of the form

$$C_{\mathcal{I}} = \left\{ (x_1, \dots, x_N) \in C_N \mid \frac{|x_i - x_j|}{|x_k - x_l|} < \frac{1}{K} \right. \\ \left. \text{if } i, j \in I_m, k \in I_s, l \in I_r, s \neq r \right\}, \quad (2.14)$$

where  $K$  is some given large number. The set contains those configurations for which the distance between points  $x_i, x_j$  for  $i, j$  from the same subset in the partition is much smaller than for different partitions<sup>4</sup>. The factorization law is then that for  $(x_1, \dots, x_N) \in C_{\mathcal{I}}$ , and for  $K$  sufficiently large, we should have

$$\mathcal{C}(x_1, \dots, x_N) = \mathcal{C}(x_{j_1}, \dots, x_{j_n}) \circ \left( \bigotimes_{i=1}^n \mathcal{C}(x_k)_{k \in I_i} \right). \quad (2.15)$$

Here, each  $j_k$  is the element of largest index in the subset  $I_k$ . The circle  $\circ$  denotes the composition of maps. In view of the infinite dimensional nature of  $V$ , one effectively assumes here, in particular, that the composition(s) be well defined. Implicit is therefore a statement about convergence as in eq. (1.5) of Thm. 2, which corresponds to  $N = 3$  and the partition  $\mathcal{I} : I_1 = \{1, 2\}, I_2 = \{3\}$ . The theorem derives its conceptual importance from this context.

Consistency conditions now arise because the above identity must hold for *all* possible partitions. In particular, by considering, for  $N = 3$  points e.g. the alternative partition  $\mathcal{J} : I_1 = \{1, 3\}, I_2 = \{2\}$ , one in effect has a kind of Jacobi-identity for the OPE. Note, however, that this identity is more subtle than for ordinary algebras, because the domains  $C_{\mathcal{I}}$  resp.  $C_{\mathcal{J}}$  in which the respective factorization identity holds, are not guaranteed to have any overlap. Both identities are rather related by analytic continuation in the points  $(x_1, x_2, x_3)$ , since the 3-point coefficient  $\mathcal{C}(x_1, x_2, x_3)$  is analytic in its arguments.

After these general points, the remainder of the paper is devoted to proving thms. 1 and 2 in a concrete (perturbative) model in  $d = 4$  dimensions, thereby lending support to the type of axiomatic framework which we have just outlined.

### 3 The flow equation framework

The model that we consider is a hermitian scalar field with self-interaction  $g\varphi^4$  and mass  $m > 0$  on flat 4-dimensional Euclidean space. The quantities of interest in this (perturbative) quantum field theory will be defined in this section via the flow equation method. Renormalization theory based on the flow equation (FE) [8, 9, 1, 2] of the renormalization

<sup>3</sup>For a single point, we put  $C(x) = id_V$ .

<sup>4</sup>The sets are reminiscent, and in fact closely related, to those defining the “little disc operad”.

group has been reviewed quite often in the literature, so we will be relatively brief. The first presentation in the form we use it here is in [14]. Reviews are in [11] and in [15] (in German).

### 3.1 Connected amputated Green functions (CAG's)

To begin, we introduce an infrared<sup>5</sup> cutoff  $\Lambda$ , and an ultraviolet cutoff  $\Lambda_0$ . These cutoffs enter the definition of the theory through the propagator  $C^{\Lambda, \Lambda_0}$ , which is defined in momentum space by

$$C^{\Lambda, \Lambda_0}(p) = \frac{1}{p^2 + m^2} \left[ \exp \left( -\frac{p^2 + m^2}{\Lambda_0^2} \right) - \exp \left( -\frac{p^2 + m^2}{\Lambda^2} \right) \right]. \quad (3.16)$$

The full propagator is recovered for  $\Lambda \rightarrow 0$  and  $\Lambda_0 \rightarrow \infty$ , and we always assume

$$0 < \Lambda, \quad \kappa := \sup(\Lambda, m) < \Lambda_0. \quad (3.17)$$

Other choices of regularization are of course admissible. The one chosen in (3.16) has the advantage of being analytic in  $p^2$  for  $\Lambda > 0$ . The propagator defines a corresponding Gaussian measure  $\mu^{\Lambda, \Lambda_0}$ , whose covariance is  $\hbar C^{\Lambda, \Lambda_0}$ . The factor of  $\hbar$  is inserted to obtain a consistent loop expansion in the following. The interaction is taken to be

$$L^{\Lambda_0}(\varphi) = \int d^4x \left( a^{\Lambda_0} \varphi(x)^2 + b^{\Lambda_0} \partial \varphi(x)^2 + c^{\Lambda_0} \varphi(x)^4 \right). \quad (3.18)$$

It contains suitable counter terms satisfying  $a^{\Lambda_0} = O(\hbar)$ ,  $b^{\Lambda_0} = O(\hbar^2)$  and  $c^{\Lambda_0} = \frac{g}{4!} + O(\hbar)$ . They will be adjusted—and actually diverge—when  $\Lambda_0 \rightarrow \infty$  in order to obtain a well defined limit of the quantities of interest to us. We have anticipated this by making them “running couplings”, i.e. functions of the ultra violet cutoff  $\Lambda_0$ . The correlation (= Schwinger- = Greens- =  $n$ -point-) functions of  $n$  basic fields with cutoff are defined by the expectation values

$$\begin{aligned} \langle \varphi(x_1) \cdots \varphi(x_n) \rangle &\equiv \mathbb{E}_{\mu^{\Lambda, \Lambda_0}} \left[ \exp \left( -\frac{1}{\hbar} L^{\Lambda_0} \right) \varphi(x_1) \cdots \varphi(x_n) \right] / Z^{\Lambda, \Lambda_0} \\ &= \int d\mu^{\Lambda, \Lambda_0} \exp \left( -\frac{1}{\hbar} L^{\Lambda_0} \right) \varphi(x_1) \cdots \varphi(x_n) / Z^{\Lambda, \Lambda_0}. \end{aligned} \quad (3.19)$$

This is just the standard Euclidean path-integral, but note that the free part in the Lagrangian has been absorbed into the Gaussian measure  $d\mu^{\Lambda, \Lambda_0}$ . The normalization factor is chosen so that  $\langle 1 \rangle = 1$ . This factor is finite only as long as we impose an additional volume cutoff. But the infinite volume limit can be taken without difficulty once we pass to perturbative connected correlation functions which we will do in a moment. For more details on this limit see [16, 11]. The path integral will be analyzed in the perturbative sense,

---

<sup>5</sup>Such a cutoff is of course not necessary in a massive theory. The IR behavior is substantially modified only for  $\Lambda$  above  $m$ .



i.e. the exponentials are expanded out and the Gaussian integrals are then performed. The full theory is obtained by sending the cutoffs  $\Lambda_0 \rightarrow \infty$  and  $\Lambda \rightarrow 0$ , for a suitable choice of the running couplings. In the FE technique, the correct behavior of the running couplings, necessary for a well-defined limit, is obtained by deriving first a differential equation for the Schwinger functions in  $\Lambda$ , and by then defining the running couplings implicitly through the boundary conditions for this equation.

These FEs are written more conveniently in terms of the hierarchy of “connected, amputated Schwinger functions” (CAG’s). Their generating functional is defined through the convolution<sup>6</sup> of the Gaussian measure with the exponentiated interaction,

$$-L^{\Lambda, \Lambda_0} := \hbar \log \mu^{\Lambda, \Lambda_0} \star \exp \left( -\frac{1}{\hbar} L^{\Lambda_0} \right) - \hbar \log Z^{\Lambda, \Lambda_0} . \quad (3.20)$$

The functional  $L^{\Lambda, \Lambda_0}$  has an expansion as a formal power series in terms of Feynman diagrams with precisely  $l$  loops,  $n$  external legs, and propagator  $C^{\Lambda, \Lambda_0}(p)$ . As the name suggests, only connected diagrams contribute, and the (free) propagators on the external legs are removed. We will not use decompositions in terms of Feynman diagrams. But we will also analyze the functional (3.20) in the sense of formal power series

$$L^{\Lambda, \Lambda_0}(\varphi) := \sum_{n>0} \sum_{l=0}^{\infty} \hbar^l \int d^4 x_1 \dots d^4 x_n \mathcal{L}_{n,l}^{\Lambda, \Lambda_0}(x_1, \dots, x_n) \varphi(x_1) \dots \varphi(x_n) , \quad (3.21)$$

where  $\varphi \in \mathcal{S}(\mathbb{R}^4)$  is any Schwartz space function. No statement is made about the convergence of the series in  $\hbar$ . The objects on the right side, the CAG’s, are the basic quantities in our analysis because they are easier to work with than the full Schwinger functions. But the latter can of course be recovered from the CAG’s.

Because the connected amputated functions in position space are translation invariant, their Fourier transforms, denoted  $\mathcal{L}_{n,l}^{\Lambda, \Lambda_0}(p_1, \dots, p_n)$ , are supported at  $p_1 + \dots + p_n = 0$ . We consequently write, by abuse of notation

$$\mathcal{L}_{n,l}^{\Lambda, \Lambda_0}(p_1, \dots, p_n) = \delta^4 \left( \sum_{i=1}^n p_i \right) \mathcal{L}_{n,l}^{\Lambda, \Lambda_0}(p_1, \dots, p_{n-1}) , \quad (3.22)$$

i.e. one of the momenta is determined in terms of the remaining  $n-1$  independent momenta by momentum conservation. It is straightforward to see that, as functions of these remaining independent momenta, the connected amputated Green functions are smooth for  $\Lambda_0 < \infty$ ,  $\mathcal{L}_{n,l}^{\Lambda, \Lambda_0}(p_1, \dots, p_{n-1}) \in C^\infty(\mathbb{R}^{4(n-1)})$ .

The FEs are obtained by taking a  $\Lambda$ -derivative of eq.(3.20):

$$\partial_\Lambda L^{\Lambda, \Lambda_0} = \frac{\hbar}{2} \left\langle \frac{\delta}{\delta \varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta \varphi} \right\rangle L^{\Lambda, \Lambda_0} - \frac{1}{2} \left\langle \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0}, \dot{C}^\Lambda \star \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0} \right\rangle + \hbar \partial_\Lambda \log Z^{\Lambda, \Lambda_0} . \quad (3.23)$$

---

<sup>6</sup>The convolution is defined in general by  $(\mu^{\Lambda, \Lambda_0} \star F)(\varphi) = \int d\mu^{\Lambda, \Lambda_0}(\varphi') F(\varphi + \varphi')$ .

Here we use the shorthand  $\dot{C}^\Lambda$  for  $\partial_\Lambda C^{\Lambda, \Lambda_0}$ , which, as we note, does not depend on  $\Lambda_0$ . By  $\langle , \rangle$  we denote the standard scalar product in  $L^2(\mathbb{R}^4, d^4x)$ , and  $\star$  denotes convolution in  $\mathbb{R}^4$ . For example

$$\langle \frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} \rangle = \int d^4x d^4y \dot{C}^\Lambda(x-y) \frac{\delta}{\delta\varphi(x)} \frac{\delta}{\delta\varphi(y)} \quad (3.24)$$

is the “functional Laplace operator”.

To define the CAG’s through the FEs, we have to impose boundary conditions. These are<sup>7</sup>, using the multi-index convention introduced above in “Notations and Conventions”:

$$\partial_{\vec{p}}^w \mathcal{L}_{n,l}^{0,\Lambda_0}(\vec{0}) = \delta_{w,0} \delta_{n,4} \delta_{l,0} \frac{g}{4!} \quad \text{for } n + |w| \leq 4, \quad (3.25)$$

as well as

$$\partial_{\vec{p}}^w \mathcal{L}_{n,l}^{\Lambda_0,\Lambda_0}(\vec{p}) = 0 \quad \text{for } n + |w| > 4. \quad (3.26)$$

The CAG’s are then determined by integrating the FE’s subject to these boundary conditions, see e.g. [14, 11].

### 3.2 Insertions of composite fields

For the purposes of this paper, and also in many applications, one would like to define not only Schwinger functions of products of the basic field, but also ones containing composite operators. These are obtained by replacing the action  $L^{\Lambda_0}$  with an action containing additional sources, expressed through smooth functionals. Particular examples of such functionals are *local* ones. Any such local functional can by definition be written as

$$F(\varphi) = \sum_A \int d^4x \mathcal{O}_A(x) f^A(x), \quad f^A \in C_0^\infty(\mathbb{R}^4), \quad (3.27)$$

where  $\mathcal{O}_A$  are composite operators as in eq. (1.3) and where the sum is finite. To simplify our discussion below, we will restrict attention to composite fields (1.2) with an even number of factors<sup>8</sup> of  $\varphi$ . We now consider instead of  $L^{\Lambda_0}$  a modified action containing sources  $f^A$ , given by replacing

$$L^{\Lambda_0} \rightarrow L_F^{\Lambda_0} := L^{\Lambda_0} + F + \sum_{j=0}^{\infty} B_j^{\Lambda_0} \underbrace{(F \otimes \cdots \otimes F)}_j, \quad (3.28)$$

---

<sup>7</sup>We restrict to BPHZ renormalization conditions in their simplest form, more general choices are of course equally admissible.

<sup>8</sup>Note that these are the fields which are invariant under the  $\mathbb{Z}_2$ -symmetry  $\varphi \rightarrow -\varphi$ . Hence the OPE closes in this sector.

where the last term represents the counter terms and is for each  $j$  a suitable linear functional<sup>9</sup>

$$B_j^{\Lambda_0} : [C^\infty(\mathcal{S}(\mathbb{R}^4))]^{\otimes j} \rightarrow C^\infty(\mathcal{S}(\mathbb{R}^4)) , \quad (3.29)$$

that is symmetric, and of order  $O(\hbar)$ . These counter terms are designed to eliminate the divergences arising from composite field insertions in the Schwinger functions when one takes  $\Lambda_0 \rightarrow \infty$ . The Schwinger functions with insertions of  $r$  composite operators are defined with the aid of functional derivatives with respect to the sources, setting the sources  $f^{A_i} = 0$  afterwards:

$$\langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_r}(x_r) \rangle := \hbar^r \frac{\delta^r}{\delta f^{A_1}(x_1) \cdots \delta f^{A_r}(x_r)} (Z^{\Lambda, \Lambda_0})^{-1} \int d\mu^{\Lambda, \Lambda_0} \exp \left( -\frac{1}{\hbar} L_F^{\Lambda_0}(\varphi) \right) \Big|_{f^{A_i}=0} . \quad (3.30)$$

The previous definition of the CAG's is a special case of this; there we take  $F = \int d^4x f(x) \varphi(x)$ , and we have  $B_j^{\Lambda_0}(F^{\otimes j}) = 0$ , because no extra counter terms are required for this simple insertion. As above, we can define a corresponding effective action as

$$-L_F^{\Lambda, \Lambda_0} := \hbar \log \mu^{\Lambda, \Lambda_0} \star \exp \left( -\frac{1}{\hbar} (L^{\Lambda_0} + F + \sum_{j=0}^{\infty} B_j^{\Lambda_0}(F^{\otimes j})) \right) - \log Z^{\Lambda, \Lambda_0} \quad (3.31)$$

which is now a functional of the sources  $f^{A_i}$ , as well as of  $\varphi$ . Differentiating  $r$  times with respect to the sources, and setting them to zero afterwards, gives the generating functionals of the CAG's with  $r$  operator insertions, namely:

$$L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}(x_1) \otimes \cdots \otimes \mathcal{O}_{A_r}(x_r)) = \frac{\delta^r L_F^{\Lambda, \Lambda_0}}{\delta f^{A_1}(x_1) \cdots \delta f^{A_r}(x_r)} \Big|_{f^{A_i}=0} . \quad (3.32)$$

The CAG's with insertions satisfy a number of obvious properties, e.g. they are multi-linear—as indicated by the tensor product notation—and symmetric in the insertions.

As the CAG's without insertions, the CAG's with insertions can be further expanded in  $\varphi$  and  $\hbar$ , and this is denoted as

$$L^{\Lambda, \Lambda_0} \left( \otimes_{i=1}^r \mathcal{O}_{A_i}(x_i) \right) = \sum_{n, l \geq 0} \hbar^l \int d^4p_1 \cdots d^4p_n \mathcal{L}_{n, l}^{\Lambda, \Lambda_0} \left( \otimes_{i=1}^r \mathcal{O}_{A_i}(x_i); p_1, \dots, p_n \right) \prod_{j=1}^n \varphi(p_j) .$$

Due to the insertions in  $\mathcal{L}_{n, l}^{\Lambda, \Lambda_0}(\otimes_j \mathcal{O}_{A_j}(x_j), \vec{p})$ , there is no restriction on the momentum set  $\vec{p}$ . However it follows from translation invariance that functions with insertions at a translated set of points  $x_j + y$  are obtained from those at  $y = 0$  upon multiplication by  $e^{iy \sum_{i=1}^n p_i}$ , i.e.

$$\mathcal{L}_{n, l}^{\Lambda, \Lambda_0} \left( \otimes_{i=1}^r \mathcal{O}_{A_i}(x_i + y); p_1, \dots, p_n \right) = e^{iy \sum_{i=1}^n p_i} \mathcal{L}_{n, l}^{\Lambda, \Lambda_0} \left( \otimes_{i=1}^r \mathcal{O}_{A_i}(x_i); p_1, \dots, p_n \right) . \quad (3.33)$$

---

<sup>9</sup> $C^\infty(\mathcal{S}(\mathbb{R}^4))$  denotes the space of smooth (in the Frechet sense) functionals. All our functionals are actually formal power series in  $\hbar$ , so we should write more accurately  $C^\infty(\mathcal{S}(\mathbb{R}^4))[[\hbar]]$  for the space appearing below.

Also, since we only consider composite fields with even powers of  $\varphi$ , only moments with  $n$  an even number are non-zero. The CAG's with insertions satisfy a FE similar in nature to the one for CAG's without insertions:

$$\begin{aligned} \partial_\Lambda L^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) &= \frac{\hbar}{2} \left\langle \frac{\delta}{\delta \varphi}, \dot{C} \star \frac{\delta}{\delta \varphi} \right\rangle L^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) \\ &- \sum_{I_1 \cup I_2 = \{1, \dots, N\}} \left\langle \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0}(\otimes_{i \in I_1} \mathcal{O}_{A_i}), \dot{C} \star \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0}(\otimes_{j \in I_2} \mathcal{O}_{A_j}) \right\rangle, \end{aligned} \quad (3.34)$$

Here we suppressed the coordinate space variables  $(x_1, \dots, x_N)$  and simply wrote  $\mathcal{O}_{A_i}$  instead of  $\mathcal{O}_{A_i}(x_i)$ . We will often use this convention in the following for the sake of brevity. When expanded out in  $\varphi$ , this equation can be expressed in momentum space as

$$\begin{aligned} \partial_\Lambda \mathcal{L}_{2n, l}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}; p_1, \dots, p_{2n}) &= \binom{2n+2}{2} \int_k \dot{C}^\Lambda(k) \mathcal{L}_{2n+2, l-1}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}; k, -k, p_1, \dots, p_{2n}) \\ &- 4 \sum_{\substack{l_1 + l_2 = l \\ n_1 + n_2 = n+1}} n_1 n_2 \mathbb{S} \left[ \mathcal{L}_{2n_1, l_1}^{\Lambda, \Lambda_0}(\otimes_{i=1}^{n_1} \mathcal{O}_{A_i}; q, p_1, \dots, p_{2n_1-1}) \dot{C}^\Lambda(q) \mathcal{L}_{2n_2, l_2}^{\Lambda, \Lambda_0}(p_{2n_1}, \dots, p_{2n}) \right. \\ &+ \left. \sum_{\substack{I_1 \cup I_2 = \{1, \dots, N\} \\ I_1 \neq \emptyset \neq I_2}} \int_k \mathcal{L}_{2n_1, l_1}^{\Lambda, \Lambda_0}(\otimes_{i \in I_1} \mathcal{O}_{A_i}; k, p_1, \dots, p_{2n_1-1}) \dot{C}^\Lambda(k) \mathcal{L}_{2n_2, l_2}^{\Lambda, \Lambda_0}(\otimes_{j \in I_2} \mathcal{O}_{A_j}; -k, p_{2n_1}, \dots, p_{2n}) \right] \end{aligned} \quad (3.35)$$

with  $q = p_{2n_1} + \dots + p_{2n}$  and where  $\mathbb{S}$  is the symmetrization operator acting on functions of the momenta  $(p_1, \dots, p_{2n})$  by taking the mean value over all permutations  $\pi$  of  $1, \dots, 2n$  satisfying  $\pi(1) < \pi(2) < \dots < \pi(2n_1 - 1)$  and  $\pi(2n_1) < \dots < \pi(2n)$ . It is crucial to note that the FE for the CAG's with  $N \geq 2$  insertions is not linear homogeneous, but involves a “source term” which is quadratic in the CAG's with less than  $N$  insertions. If we want to integrate the FEs with insertions, we therefore have to ascent in the number of insertions. The CAG's with insertions are thereby uniquely defined once we impose suitable boundary conditions on the corresponding FE. The simplest choice in the case of  $N \geq 2$  insertions is

$$\partial_{\vec{p}}^w \mathcal{L}_{n, l}^{\Lambda_0, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i); \vec{p}) = 0 \quad \text{for all } w, n, l. \quad (3.36)$$

For CAG's with one insertion we choose (“normal ordering”)

$$\partial_{\vec{p}}^w \mathcal{L}_{n, l}^{0, \Lambda_0}(\mathcal{O}_A(0); \vec{0}) = i^{|w|} w! \delta_{w, w'} \delta_{n, n'} \delta_{l, 0} \quad \text{for } n + |w| \leq [A] \quad (3.37)$$

$$\partial_{\vec{p}}^w \mathcal{L}_{n, l}^{\Lambda_0, \Lambda_0}(\mathcal{O}_A(0); \vec{p}) = 0 \quad \text{for } n + |w| > [A] \quad . \quad (3.38)$$

### 3.3 Regularized CAG's with $N \geq 2$ insertions

It is not hard to see that, as long as we keep the UV cutoff  $\Lambda_0$  finite, the CAG's with insertions depend smoothly on the points  $x_1, \dots, x_N$ , as well as on the momenta  $p_1, \dots, p_n$ . In the limit as  $\Lambda_0 \rightarrow \infty$ , smoothness in the  $x_i$ 's however is lost, and the CAG's develop singularities for configurations such that some of the points  $x_i$  coincide. This is of course not a problem, nor unexpected—the Greens functions in QFT are usually singular for coinciding points—reflecting the singular nature of the operators themselves.

One of the main technical advances of this paper, generalizing methods of [4, 17], is to consider, besides the original CAG's, certain “regularized” (sometimes called “oversubtracted”) versions thereof, which possess more regularity in the  $x_i$ 's as  $\Lambda_0 \rightarrow \infty$ , i.e. which remove, in a certain sense, some or all of the singularities at coincident points. In the case of  $N = 2$  insertions, these divergences can be removed by simply changing the boundary conditions for the CAG's [17]. These regularized CAG's are parametrized by a single integer parameter  $D \geq -1$  and defined by the same FE, eq.(3.34), but subject to the boundary conditions

$$\partial_{\vec{p}}^w \mathcal{L}_{D,n,l}^{\Lambda_0, \Lambda_0}(\mathcal{O}_{A_1}(x) \otimes \mathcal{O}_{A_2}(0); \vec{0}) = 0 \quad \text{for } n + |w| \leq D \quad (3.39)$$

as well as

$$\partial_{\vec{p}}^w \mathcal{L}_{D,n,l}^{\Lambda_0, \Lambda_0}(\mathcal{O}_{A_1}(x) \otimes \mathcal{O}_{A_2}(0); \vec{p}) = 0 \quad \text{for } n + |w| > D \quad (3.40)$$

instead of eq.(3.36). It has been shown in [17] that these functions are of differentiability class  $C^{D-[A_1]-[A_2]}$  in  $x$ . This justifies the interpretation of  $D$  as a regularization parameter. For more than two insertions, one has a choice for which subset of the arguments  $x_1, \dots, x_N$  one wants to remove the singularities, and up to what degree. In order to specify this, we consider all subsets  $I \subset \{1, \dots, N\}$  with  $|I| \geq 2$  and define the corresponding “partial diagonal” by

$$\text{Diag}_I = \{(x_1, \dots, x_N) \in (\mathbb{R}^4)^N \mid x_i = x_j \text{ for } i, j \in I\}. \quad (3.41)$$

The CAG's with insertions  $L^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i))$ , or the versions thereof which are expanded in  $\varphi$ , see eq. (3.35), develop singularities at those diagonals in the limit as  $\Lambda_0 \rightarrow \infty$ . Next, we specify a collection  $\mathcal{I}$  of subsets of  $\{1, \dots, N\}$ . We agree that  $\{1, \dots, N\}$  is always in  $\mathcal{I}$ . For each corresponding partial diagonal an integer  $D_I \geq -1$  is assigned which specifies the degree of regularization of the CAG at that diagonal. The tuple of all such numbers is denoted

$$\mathbf{D} = (D_I \mid I \in \mathcal{I}) \in (\mathbb{Z}_{\geq -1})^{|\mathcal{I}|} \quad (3.42)$$

and given such a tuple, one defines the corresponding regulated CAG's by the FE:

$$\begin{aligned} \partial_{\Lambda} L_{\mathbf{D}}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) &= \frac{\hbar}{2} \left\langle \frac{\delta}{\delta \varphi}, \dot{C} \star \frac{\delta}{\delta \varphi} \right\rangle L_{\mathbf{D}}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) - \left\langle \frac{\delta}{\delta \varphi} L_{\mathbf{D}}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}), \dot{C} \star \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0} \right\rangle \\ &\quad - \sum_{\substack{I_1 \cup I_2 = \{1, \dots, N\} \\ I_i \in \mathcal{I} \text{ or } |I_i| = 1}} \left\langle \frac{\delta}{\delta \varphi} L_{\mathbf{D}_1}^{\Lambda, \Lambda_0}(\otimes_{i \in I_1} \mathcal{O}_{A_i}), \dot{C} \star \frac{\delta}{\delta \varphi} L_{\mathbf{D}_2}^{\Lambda, \Lambda_0}(\otimes_{j \in I_2} \mathcal{O}_{A_j}) \right\rangle. \end{aligned} \quad (3.43)$$

Here, we have set  $\mathbf{D}_i = \{D_J \mid J \in \mathcal{I}, J \subseteq I_i\}$  for  $i = 1, 2$ . When  $|I_1| = 1$  (or when  $|I_2| = 1$ ) then the corresponding  $D$ -symbol on  $L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_i})$  is absent—the 1-point functions are already smooth in  $x_i$ . The boundary conditions are taken as

$$\partial_{\vec{p}}^w \mathcal{L}_{\mathbf{D}}^{0, \Lambda_0}(\mathcal{O}_{A_1}(x_1) \otimes \cdots \otimes \mathcal{O}_{A_N}(0); \vec{0}) = 0 \quad \text{for } n + |w| \leq D_{\{1, \dots, N\}} \quad (3.44)$$

and

$$\partial_{\vec{p}}^w \mathcal{L}_{\mathbf{D}}^{\Lambda_0, \Lambda_0}(\mathcal{O}_{A_1}(x_1) \otimes \cdots \otimes \mathcal{O}_{A_N}(0); \vec{p}) = 0 \quad \text{for } n + |w| > D_{\{1, \dots, N\}}. \quad (3.45)$$

Our definition of the regulated CAG's is recursive in nature; first those with  $N = 2$  insertions are defined, then those with  $N = 3$ , etc.

The regularized multi-point CAG's are the main new technical tool in this paper. For  $N = 2$ , we only have to specify the value of  $D_{\{1, 2\}}$ , as  $I = \{1, 2\}$  is the only subset of  $\{1, 2\}$  with at least 2 elements. In that case, the definition of the regularized two-point CAG hence depends on the specification of only one number, and it reduces to the one already given above in eqs. (3.39) and (3.40). In this paper, we will only need additionally the case of  $N = 3$  points. In that case,  $\mathcal{I}$  may be any collection of the subsets  $I = \{1, 2, 3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}$ , and the corresponding numbers  $D_I$  control the degree of regularity of the CAG  $L_{\mathbf{D}}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}(x_1) \otimes \mathcal{O}_{A_2}(x_2) \otimes \mathcal{O}_{A_3}(x_3))$  at the respective diagonals  $\{x_1 = x_2 = x_3\}, \{x_1 = x_2\}, \{x_2 = x_3\}, \{x_1 = x_3\}$ . Of particular interest to us will be the following special cases:

- The collection  $\mathcal{I} = \{\{1, 2, 3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}$  together with the choice  $D_{\{1, 2, 3\}} = D_{\{1, 2\}} = D_{\{1, 3\}} = D_{\{2, 3\}} = -1$  corresponds to no regularization at all. We will denote these CAG's without regularization as before simply by  $L^{\Lambda, \Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i})$ .
- The collection  $\mathcal{I} = \{\{1, 2, 3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}$  together with the choice  $D_I = \sum_{i \in I} [A_i]$  for all  $I \in \mathcal{I}$  can be shown to be regular (continuous) at all partial diagonals, uniformly as  $\Lambda_0 \rightarrow \infty$ .
- The collection  $\mathcal{I} = \{\{1, 2, 3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}$  together with the choice  $D_{\{1, 2, 3\}} = D_{\{1, 2\}} = D_{\{1, 3\}} = -1, D_{\{2, 3\}} > -1$  corresponds to no regularization at all except at the diagonal  $\{x_2 = x_3\}$ . These objects appear when analysing the OPE of two fields localized at  $x_2, x_3$  in a Schwinger function of three fields, with the field at  $x_1$  playing the role of a “spectator”.
- It is not necessary that  $\mathcal{I}$  contains all subsets of  $\{1, 2, 3\}$  (of order  $> 1$ ), as in the above three examples. The sum in the defining FE then has fewer terms, as described above. We will also consider these objects.

As we will later be interested in spacetime derivatives of the CAG's, the following relations will be useful. They are generalizations of the *Lowenstein rules*, given e.g. in [18, 17]. For  $N = 1$  insertion, the rule is simply that

$$\partial_x^w L^{\Lambda, \Lambda_0}(\mathcal{O}_A(x)) = L^{\Lambda, \Lambda_0}(\partial_x^w \mathcal{O}_A(x)) \quad (3.46)$$

where  $\partial_x^w$  is a multiple partial derivative acting on  $x$ , see our notations and conventions for more on multi-index notation. Note that the identity is (slightly) non-trivial, because the  $\partial_x^w$  symbol on the left side acts on the CAG, while on the right side it acts on the operator  $\mathcal{O}_A$ , i.e. a monomial in  $\varphi$  and its derivatives, which is then inserted into a CAG. So, a priori, both sides mean potentially different things, but they are in fact equal. A generalization also holds for CAG's with multiple insertions.

$$\partial_{x_j}^w L_{\mathbf{D}}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) = L_{\mathbf{D}}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}(x_1) \otimes \cdots \partial_{x_j}^w \mathcal{O}_{A_j}(x_j) \otimes \cdots \mathcal{O}_{A_N}(x_N)) \quad (3.47)$$

Derivatives in directions of the “center of mass” on the total diagonal, or a partial diagonal associated with a subset  $I \subset \{1, \dots, N\}$ , have a special status. For a multiindex  $v$  such a derivative is given by  $(\sum_{i \in I} \partial_{x_i})^v$

$$(\sum_{j \in I} \partial_{x_j})^v L_{\mathbf{D}}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) = \sum_{\sum_{i \in I} w_i = v} c_{\{w_i\}} L_{\mathbf{D}(I, \{w_i\})}^{\Lambda, \Lambda_0}(\prod_{i \in I} \partial_{x_i}^{w_i} \otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) . \quad (3.48)$$

Here  $\mathbf{D}(I, \{w_i\}) = \{D'_J \mid J \in \mathcal{I}\}$  is defined as

$$D'_J = \begin{cases} D_J & \text{if } J \text{ is not a subset of } I, \\ D_J + \sum_{j \in J} |w_j| & \text{if } J \text{ is a subset of } I. \end{cases} \quad (3.49)$$

The symbols  $c_{\{w_i\}}$  are binomial type coefficients, see our notations and conventions section. The last condition is most straightforward to understand when  $I = \{1, \dots, N\}$ . The point is that the CAG is always smooth in the center of mass variable  $x_1 + \cdots + x_N$ , so derivatives with respect to this variable never make the divergences of the CAG's worse. The degree of regularity of the right side is hence the same as that without the derivatives taken, which, due to the higher dimensions of the operator insertions, results in a higher regularity degree  $D_{\{1, \dots, N\}}$  on the total diagonal.

All of the generalized Lowenstein rules, although complicated looking, can be proved via a simple but powerful *principle in the FE method*, which we will use several times over in this article. The general principle simply says that if we have two hierarchies of functions satisfying the same *FE* and same *boundary condition*, then they must coincide. In the case of the Lowenstein identities, it is easy to find what FE's each side has to satisfy, and what boundary conditions. For each identity, these are easily seen to coincide, thus proving the identity.

### 3.4 Amputated Greens functions (AG's) with insertions

Although the connected amputated Greens functions (CAG's) are the basic building blocks of the correlation functions which will be mostly considered, we will occasionally also need the non-connected version of these, called AG's. Their definition is

$$G^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) = \sum_{\alpha=1}^N (-1)^{\alpha+1} \sum_{I_1 \cup \dots \cup I_\alpha = \{1, \dots, N\}} \prod_{i=1}^{\alpha} \hbar^{N-\alpha} L^{\Lambda, \Lambda_0}(\otimes_{j \in I_i} \mathcal{O}_{A_j}) \quad (3.50)$$

As usual, we also consider the expanded quantities in  $\hbar$  and  $\varphi$ ; these are denoted in the present case as  $\mathcal{G}_{n,l}^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}, \vec{p})$ , where as usual,  $l$  indicates the power of  $\hbar$ , and  $n$  the power of  $\varphi$ . As the name suggests, these are the amputated versions of the Schwinger (=Greens) functions,

$$\begin{aligned} & \left\langle \prod_{i=1}^N \mathcal{O}_{A_i}(x_i) \prod_{j=1}^n \hat{\varphi}(p_j) \right\rangle_l \text{ loops} \prod_{k=1}^n (C^{\Lambda,\Lambda_0}(p_k))^{-1} \\ &= \sum_{j=1}^n (-1)^{j+1} \sum_{\substack{I_1 \cup \dots \cup I_j = \{1, \dots, n\} \\ l_1 + \dots + l_j = l}} \hbar^{n+l+1-j} \mathcal{G}_{|I_1|,l_1}^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i), \vec{p}_{I_1}) \bar{\mathcal{L}}_{|I_2|,l_2}^{\Lambda,\Lambda_0}(\vec{p}_{I_2}) \cdots \bar{\mathcal{L}}_{|I_j|,l_j}^{\Lambda,\Lambda_0}(\vec{p}_{I_j}) \end{aligned} \quad (3.51)$$

where  $\bar{\mathcal{L}}_{n,l}^{\Lambda,\Lambda_0}$  are the expansion coefficients of the generating functional  $\bar{L}^{\Lambda,\Lambda_0}(\varphi) = L^{\Lambda,\Lambda_0}(\varphi) + \frac{1}{2} \langle \varphi, (C^{\Lambda,\Lambda_0})^{-1} \star \varphi \rangle$  without the momentum conservation delta functions taken out. We will use this relation later.

By contrast to the CAG's, the AG's satisfy linear *homogeneous* FE's which are

$$\partial_\Lambda G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) = \frac{\hbar}{2} \left\langle \frac{\delta}{\delta \varphi}, \dot{C} \star \frac{\delta}{\delta \varphi} \right\rangle G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) - \frac{\hbar}{2} \left\langle \frac{\delta}{\delta \varphi} G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}), \dot{C} \star \frac{\delta}{\delta \varphi} L^{\Lambda,\Lambda_0} \right\rangle. \quad (3.52)$$

The fact that the AG's satisfy a homogeneous FE is a welcome simplification, which is unfortunately counterbalanced by the fact that the boundary conditions for the AG's are more non-trivial. Therefore, as a compromise between simple FE and simple boundary conditions, we will not work with the full AG's in the following, but instead define the slightly modified objects

$$F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) := G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) + (-1)^N \prod_{i=1}^N L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}). \quad (3.53)$$

Using the definitions of the CAG's given above, these functionals are seen to obey the FE

$$\begin{aligned} & \partial_\Lambda F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) \\ &= \frac{\hbar}{2} \left\langle \frac{\delta}{\delta \varphi}, \dot{C} \star \frac{\delta}{\delta \varphi} \right\rangle F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) - \frac{\hbar}{2} \left\langle \frac{\delta}{\delta \varphi} F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}), \dot{C} \star \frac{\delta}{\delta \varphi} L^{\Lambda,\Lambda_0} \right\rangle \\ & - (-1)^N \hbar \sum_{i \neq j \in \{1, \dots, N\}} \left\langle \frac{\delta}{\delta \varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}), \dot{C} \star \frac{\delta}{\delta \varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j}) \right\rangle \prod_{r \in \{1, \dots, N\} \setminus \{i, j\}} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r}) \end{aligned} \quad (3.54)$$

and the trivial boundary conditions

$$\partial_{\vec{p}}^w \mathcal{F}_{n,l}^{\Lambda_0,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}; \vec{p}) = 0 \quad \text{for all } n, l, w, \quad (3.55)$$



with a calligraphic letter  $\mathcal{F}_{n,l}^{\Lambda,\Lambda_0}$  denoting as usual the objects appearing in the expansion of  $F^{\Lambda,\Lambda_0}$  in powers of  $\hbar, \varphi$ . Like the CAG's with multiple insertions, the  $F$  functionals are divergent on the partial diagonals, i.e. whenever two or more spacetime arguments coincide. To analyze the OPE we also need regularized  $F$  functionals, which will be called  $F_D$ . They are defined to satisfy the same FE as  $F$ , eq.(3.54), but the boundary conditions are set to be

$$\partial_{\vec{p}}^w \mathcal{F}_{n,l,D}^{0,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}; \vec{0}) \Big|_{x_N=0} = 0 \quad \text{for } n + |w| \leq D \quad (3.56)$$

$$\partial_{\vec{p}}^w \mathcal{F}_{n,l,D}^{\Lambda_0,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}; \vec{p}) \Big|_{x_N=0} = 0 \quad \text{for } n + |w| > D. \quad (3.57)$$

Note that in the  $N = 2$  case  $F_D$  reduces to the (regularized) bilocal insertions, i.e.

$$F_D^{\Lambda,\Lambda_0}(\mathcal{O}_A(x) \otimes \mathcal{O}_B(0)) = \hbar L_D^{\Lambda,\Lambda_0}(\mathcal{O}_A(x) \otimes \mathcal{O}_B(0)) \quad (3.58)$$

since both share the same FE and boundary conditions. For  $N \geq 3$ , however, such a simple relation does not exist. We also define the complete regularized AG's as

$$G_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) := F_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) + (-1)^N \prod_{i=1}^N L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}) \quad (3.59)$$

since the CAG's with single insertion are smooth without any regularization.

### 3.5 OPE coefficients

We next give the definition of the OPE coefficients. To have more compact formulas, let us define the operator  $\mathcal{D}^A$  acting on differentiable functionals  $F(\varphi)$  of Schwartz space functions  $\varphi \in \mathcal{S}(\mathbb{R}^4)$  by

$$\mathcal{D}^A F(\varphi) = \frac{1}{n! w!} \partial_{\vec{p}}^w \frac{\delta^n}{\delta \hat{\varphi}(p_1) \cdots \delta \hat{\varphi}(p_n)} F(\varphi) \Big|_{\hat{\varphi}=0, \vec{p}=0}, \quad (3.60)$$

where  $A = \{n, w\}$ . Further, let us also define the multivariate Taylor expansion operator through

$$\mathbb{T}_{\vec{x} \rightarrow \vec{y}}^j f(\vec{x}) = \mathbb{T}_{(x_1, \dots, x_N) \rightarrow (y_1, \dots, y_N)}^j f(x_1, \dots, x_N) = \sum_{|w|=j} \frac{(\vec{x} - \vec{y})^w}{w!} \partial^w f(\vec{y}) \quad (3.61)$$

where  $\vec{x} = (x_1, \dots, x_N)$  and where  $f$  is a sufficiently smooth function on  $\mathbb{R}^{4N}$ . For expansions around zero will use the shorthand  $\mathbb{T}_{\vec{x} \rightarrow \vec{0}}^j =: \mathbb{T}_{\vec{x}}^j$ . Then the OPE coefficients are defined as follows:

**Definition 1.** Let  $\Delta := [C] - ([A_1] + \dots + [A_N])$ . Then we define the OPE coefficients by

$$\mathcal{C}_{A_1 \dots A_N}^C(x_1, \dots, x_{N-1}, 0) := \mathcal{D}^C \left\{ G_{[C]-1}^{0, \Lambda_0} \left( \left( 1 - \sum_{j < \Delta} \mathbb{T}_x^j \right) \otimes_{i=1}^N \mathcal{O}_{A_i}(x_i) \right) \right\}, \quad (3.62)$$

where it is understood that  $x_N = 0$ .

**Remark:** In the case  $N = 2$  this definition is equivalent to the one given in [4]. Note also that the OPE coefficients are translation invariant, so we may e.g. put the last point to zero by a translation, as we have done above to get a simpler formula.

## 4 Bounds on Green's functions with insertions

In the previous section, we have introduced the quantities of interest in this paper, the (connected, amputated) Schwinger functions, and the OPE coefficients. In order to prove thm. 1 and thm. 2, we will need suitable bounds for these objects. These bounds will be presented and proved in the following sections. Our bounds are obtained using an inductive scheme based on the renormalization group FEs, relying heavily on the bounds already established in [4] and also [15, 18, 19]. For the sake of brevity we will refer the reader to the existing papers for technical details whenever possible. In the following we will also set  $\hbar = 1$ .

### 4.1 CAG's with up to one insertion

Bounds on CAG's without insertions and on those with one insertion were derived in [19, 4]. These bounds are a crucial input for the subsequent bounds on CAG's with multiple insertions, because the CAG's with one insertion enter the FE for those with two insertions as “inhomogeneities” [cf. eq. (3.34)], those with up to two insertions enter the FE for those with three etc. In this way, many of the estimates will be seen to carry over from the case of one insertion in that way.

Let us recall the bound for the CAG's without insertions first [19, 4]. There exists a constant  $K > 0$  such that for  $2n + |w| \geq 5$  (recall also the definitions of  $|\vec{p}|$  and  $\kappa$  from our notations and conventions section above)

$$|\partial_{\vec{p}}^w \mathcal{L}_{2n,l}^{\Lambda, \Lambda_0}(p_1, \dots, p_{n-1})| \leq \sqrt{|w|!} \Lambda^{4-2n-|w|} K^{(2n+4l-4)(|w|+1)} (n+l-2)! \sum_{\hat{a}=0}^{\ell(n,l)} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!}, \quad (4.63)$$

where  $\ell(n, l) = l$  if  $n \geq 2$  and  $\ell(n, l) = l - 1$  if  $n = 1$ . For  $2n + |w| \leq 4$  one has the estimates

$$|\mathcal{L}_{4,l}^{\Lambda, \Lambda_0}(\vec{p})| \leq \frac{K^{2l}}{(l+1)^2 2^4} (1+l)! \sum_{\hat{a}=0}^l \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!}, \quad (4.64)$$

$$|\partial_{\vec{p}}^w \mathcal{L}_{2,l}^{\Lambda, \Lambda_0}(p)| \leq \sup(|p|, \kappa)^{2-|w|} \frac{K^{2l-1}}{(l+1)^2} l! \sum_{\lambda=0}^{l-1} \frac{\log^\lambda(\sup(\frac{|p|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!}. \quad (4.65)$$

We also recall the following bound for the CAG's with one insertion [4]. Fix any  $A = \{r, v\}$ . Then

$$\begin{aligned} |\partial_{\vec{p}}^w \mathcal{L}_{2n,l}^{\Lambda, \Lambda_0}(\mathcal{O}_A(0); \vec{p})| &\leq \Lambda^{[A]-2n-|w|} K^{(4n+8l-4)|w|} K^{[A](n+2l)^3} \sqrt{|w|!|v|!} \\ &\times \sum_{\mu=0}^{d(N=1,n,l,w,[A])} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^\mu \sum_{\lambda=0}^{2l+n-1} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!}, \end{aligned} \quad (4.66)$$

where  $K > 0$  is some constant not depending on  $n$  and  $l$ , and where we defined  $d(N, n, l, w, D') := 2D'(n + l + 2(N - 1)) + \sup(D' + 1 - 2n - |w|, 0)$ . Eventually we would of course like to remove the cutoffs, i.e. take the limits  $\Lambda \rightarrow 0$  and  $\Lambda_0 \rightarrow \infty$ . In this respect, the following bounds, which hold for  $\Lambda \leq m$ , will be useful [4]

$$\begin{aligned} |\partial_{\vec{p}}^w \mathcal{L}_{2n,l}^{\Lambda, \Lambda_0}(\vec{p})| &\leq m^{4-2n-|w|} \frac{K^{(2n+4l-4)(|w|+1)}}{n!} (n + l - 1)! \\ &\times \sqrt{|w|! (|w| + 2n - 4)!} \sum_{\lambda=0}^l \frac{\log_+^\lambda(\frac{|\vec{p}|}{m})}{2^\lambda \lambda!} \quad \text{for } 2n + |w| \geq 5 \end{aligned} \quad (4.67)$$

$$\begin{aligned} |\partial_{\vec{p}}^w \mathcal{L}_{2n,l}^{\Lambda, \Lambda_0}(\mathcal{O}_A(0); \vec{p})| &\leq m^{[A]-2n-|w|} K^{(4n+8l-4)|w|} K^{[A](n+2l)^3} \sqrt{|w|!|v|!} \\ &\times \sqrt{[2n + |w| - [A]]_+!} \sum_{\mu=0}^{d(N=1,n,l,w,[A])} \left(\frac{|\vec{p}|}{m}\right)^\mu \sum_{\lambda=0}^{2l+n-1} \frac{\log_+^\lambda(\frac{|\vec{p}|}{m})}{2^\lambda \lambda!}, \end{aligned} \quad (4.68)$$

where by  $[\cdot]_+$  we mean the positive part of the respective expression.

## 4.2 CAG's with two insertions

Our first new estimate is a generalization of the bound on the CAG's with two insertions given in thm. 3 of [4].

**Theorem 3.** *For any  $D \leq D' = [A_1] + [A_2]$  there exists a constant  $K > 0$  such that*

$$\begin{aligned} |\partial_{\vec{p}}^w \mathcal{L}_{D,2n,l}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}(x) \otimes \mathcal{O}_{A_2}(0); \vec{p})| &\leq \Lambda^{D-2n-|w|} K^{(4n+8l-3)(|w|+D'-D)} K^{D'(n+2l)^3} \sqrt{|v_1|!|v_2|!} \\ &\times \frac{\sqrt{(|w| + D' - D)!}}{|x|^{D'-D}} \sum_{\mu=0}^{d(2,n,l,w,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^\mu \sum_{\lambda=0}^{2l+n} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!} \end{aligned} \quad (4.69)$$

with  $d(N, n, l, w, D') := 2D'(n + l + 2(N - 1)) + \sup(D' + 1 - 2n - |w|, 0)$ . Here  $v_i$  refers to the multi-indices in  $A_i \equiv \{n_i, v_i\}$ .

**Remarks:**

1. We can see explicitly that the parameter  $D$  improves regularity at  $x = 0$ .
2. The CAG's with two insertions have been bounded in [4] for the particular choice  $D = [A_1] + [A_2]$  (full regularization) and  $|w| \leq D + 1$ .

*Proof.* The strategy is to integrate the differentiated FE

$$\begin{aligned}
& \partial_\Lambda \partial_{\vec{p}}^w \mathcal{L}_{D,2n,l}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}; p_1, \dots, p_{2n}) = \\
& = \binom{2n+2}{2} \int_k \dot{C}^\Lambda(k) \partial_{\vec{p}}^w \mathcal{L}_{D,2n+2,l-1}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}; k, -k, p_1, \dots, p_{2n}) - 4 \sum_{\substack{l_1+l_2=l \\ n_1+n_2=n+1}} n_1 n_2 \\
& \times \mathbb{S} \left[ \sum_{w_1+w_2+w_3=w} c_{\{w_j\}} \partial_{\vec{p}}^{w_1} \mathcal{L}_{D,2n_1,l_1}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}; q, p_1, \dots, p_{2n_1-1}) \partial_{\vec{p}}^{w_2} \dot{C}^\Lambda(q) \partial_{\vec{p}}^{w_3} \mathcal{L}_{2n_2,l_2}^{\Lambda,\Lambda_0}(p_{2n_1}, \dots, p_{2n}) \right. \\
& \left. + \sum_{w_1+w_2=w} c_{\{w_j\}} \int_k \partial_{\vec{p}}^{w_1} \mathcal{L}_{2n_1,l_1}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_1}; k, p_1, \dots, p_{2n_1-1}) \dot{C}^\Lambda(k) \partial_{\vec{p}}^{w_2} \mathcal{L}_{2n_2,l_2}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_2}; -k, p_{2n_1}, \dots, p_{2n}) \right] \\
& \tag{4.70}
\end{aligned}$$

over  $\Lambda$  and bound each term on the right hand side separately. For the first two terms on the right hand side of this equation, the bound is verified to hold inductively as one goes up in  $n + l$  and for fixed  $n + l$  goes up in  $l$ . This general procedure is very similar to the one employed in [4]. To bound the expression in the last line of eq.(4.70), we will make use of the known bounds on the CAG's with one insertion, (4.66).

When integrating eq.(4.70) over  $\Lambda$  we have to distinguish three cases:

- (A) Contributions with  $2n + |w| > D$  are referred to as *irrelevant*. Here the boundary conditions are given at  $\Lambda = \Lambda_0$ , see eq.(3.40). Therefore, we integrate over  $\Lambda'$  from  $\Lambda$  to  $\Lambda_0$  in this case.
- (B) Contributions with  $2n + |w| \leq D$  are referred to as *relevant*. The boundary conditions for relevant terms, eq.(3.39), are given at  $\Lambda = 0$  and at vanishing external momentum,  $\vec{p} = \vec{0}$ . Thus, we will integrate over  $\Lambda'$  from 0 to  $\Lambda$  in this case.
- (C) Contributions with  $2n + |w| \leq D$  and  $\vec{p} \neq \vec{0}$  will be obtained from (A),(B) with the help of a Taylor expansion in  $\vec{p}$ .

**(A) Irrelevant terms ( $2n + |w| > D$ ):**

**First term on the r.h.s. of the flow equation:** Substituting our inductive bound, (4.69), and

$$\dot{C}^\Lambda(k) = -\frac{2}{\Lambda^3} e^{-\frac{k^2+m^2}{\Lambda^2}} \quad (4.71)$$

into the first term on the r.h.s. of eq.(4.70) and integrating over  $\Lambda'$  from  $\Lambda$  to  $\Lambda_0$  yields the inequality [recall the definition of  $|\vec{p}|_{2n+2}$  from eq.(1.10)]

$$\begin{aligned} & \left| \int_{\Lambda}^{\Lambda_0} d\Lambda' \binom{2n+2}{2} \int d^4k \dot{C}^{\Lambda'}(k) \partial_{\vec{p}}^w \mathcal{L}_{D,2n+2,l-1}^{\Lambda',\Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}; k, -k, p_1, \dots, p_{2n}) \right| \\ & \leq \int_{\Lambda}^{\Lambda_0} d\Lambda' \binom{2n+2}{2} \int d^4k \frac{2}{\Lambda^3} e^{-\frac{k^2+m^2}{\Lambda^2}} \Lambda'^{D-(2n+2)-|w|} K^{(4n+8l-7)(|w|+D'-D)} K^{D'(n+2l-1)^3} \\ & \times \sqrt{|v_1|!|v_2|!} \frac{\sqrt{(|w|+D'-D)!}}{|x|^{D'-D}} \sum_{\mu=0}^{d(2,n+1,l-1,w,D')} \frac{1}{\sqrt{\mu!}} \left( \frac{|\vec{p}|_{2n+2}}{\Lambda'} \right)^\mu \sum_{\lambda=0}^{2l+n-1} \frac{\log^\lambda(\sup(\frac{|\vec{p}|_{2n+2}}{\kappa'}, \frac{\kappa'}{m}))}{2^\lambda \lambda!} \\ & \leq \binom{2n+2}{2} K^{(4n+8l-7)(|w|+D'-D)} K^{D'(n+2l-1)^3} \sqrt{|v_1|!|v_2|!} \frac{\sqrt{(|w|+D'-D)!}}{|x|^{D'-D}} \sum_{\lambda=0}^{2l+n-1} \frac{2}{2^\lambda \lambda!} \\ & \times \int_{\Lambda}^{\Lambda_0} d\Lambda' \Lambda'^{D-2n-1-|w|} e^{-\frac{m^2}{\Lambda'^2}} \sum_{\mu=0}^{d(2,n,l,w,D')} \frac{1}{\sqrt{\mu!}} \int d^4\left(\frac{k}{\Lambda'}\right) \left( \frac{|\vec{p}|_{2n+2}}{\Lambda'} \right)^\mu \log^\lambda(\sup(\frac{|\vec{p}|_{2n+2}}{\kappa'}, \frac{\kappa'}{m})) e^{-\frac{k^2}{\Lambda'^2}} \end{aligned} \quad (4.72)$$

One can show [19] that the momentum integral in the last line is bounded by

$$\begin{aligned} & \sum_{\mu=0}^d \frac{1}{\sqrt{\mu!}} \int d^4(k/\Lambda') \left( \frac{|\vec{p}|_{2n+2}}{\Lambda'} \right)^\mu \log^\lambda(\sup(\frac{|\vec{p}|_{2n+2}}{\kappa'}, \frac{\kappa'}{m})) e^{-\frac{k^2}{\Lambda'^2}} \\ & \leq 2^d \sum_{\mu=0}^d \frac{1}{\sqrt{\mu!}} \left( \frac{|\vec{p}|}{\Lambda'} \right)^\mu \left( \log^\lambda(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m})) + \sqrt{\lambda!} \right) . \end{aligned} \quad (4.73)$$

The  $\Lambda'$  integral can then be estimated using the formula [19]

$$\sum_{\lambda=0}^{2l+n-1} \frac{1}{2^\lambda \lambda!} \int_{\Lambda}^{\Lambda_0} d\Lambda' \Lambda'^{-s-1} \left( \log^\lambda(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m})) + \sqrt{\lambda!} \right) \leq 5 \frac{\Lambda^{-s}}{s} \sum_{\lambda=0}^{2l+n-1} \frac{1}{2^\lambda \lambda!} \log^\lambda(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m})) \quad (4.74)$$

which holds for any  $s \in \mathbb{N}$ . Using (4.73) and (4.74) in formula (4.72) and also noting the relation  $(2l + n - 1)^3 = (2l + n)^3 - 3(2l + n)(2l + n - 1) - 1$ , we find the bound

$$\begin{aligned}
& \left| \int_{\Lambda}^{\Lambda_0} d\Lambda' \binom{2n+2}{2} \int d^4k \dot{C}^{\Lambda'}(k) \partial_{\vec{p}}^w \mathcal{L}_{D,2n+2,l-1}^{\Lambda',\Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}; k, -k, p_1, \dots, p_{2n}) \right| \\
& \leq \Lambda^{D-2n-|w|} K^{D'(2l+n)^3} K^{(4n+8l-3)(|w|+D'-D)} \frac{\sqrt{|v_1|!|v_2|!(|w|+D'-D)!}}{|x|^{D'-D}} \\
& \times \sum_{\mu=0}^{d(2,n,l,w,D')} \frac{1}{\sqrt{\mu!}} \left( \frac{|\vec{p}|}{\Lambda} \right)^{\mu} \sum_{\lambda=0}^{2l+n-1} \frac{\log^{\lambda}(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^{\lambda} \lambda!} \\
& \times K^{-3D'(2l+n)(2l+n-1)-D'-4(|w|+D'-D)} \binom{2n+2}{2} 2^{d(2,n,l,w,D')} \frac{10}{2n+|w|-D}
\end{aligned} \tag{4.75}$$

We see that this contribution satisfies the inductive bound, (4.69) multiplied by <sup>10</sup>  $1/8 \mathcal{A}$ ,

$$\mathcal{A}(D', D, n, l, w) := K^{-3(|w|+D'-D)} K^{-2D'(n+2l)(n+2l-1)-D'} \tag{4.76}$$

if  $K$  is chosen large enough such that

$$10(n+1)(2n+1) 2^{d(2,n,l,w,D')} K^{-D'(n+2l)(n+2l-1)-(|w|+D'-D)} \leq 1/8, \tag{4.77}$$

for all  $n, l$ . To see that this is indeed possible, it is helpful to note that  $l \geq 1$  in this case, since otherwise the first term on the right side of the FE is simply zero.

**Second term on the r.h.s. of the flow equation:** We integrate the second term in the FE (4.70) over  $\Lambda'$  from  $\Lambda$  to  $\Lambda_0$  and insert our inductive bound as well as the known bound

---

<sup>10</sup>This factor will be crucial below in the discussion of the relevant terms at non-zero momentum, see in particular eq.(4.111).

for the CAG's without insertions [see (4.63)] to obtain

$$\begin{aligned}
& \left| \int_{\Lambda}^{\Lambda_0} d\Lambda' \sum_{\substack{l_1+l_2=l \\ n_1+n_2=n+1 \\ w_1+w_2+w_3=w}} 4n_1n_2 c_{\{w_j\}} \right. \\
& \quad \times \partial_{\vec{p}}^{w_1} \mathcal{L}_{D,2n_1,l_1}^{\Lambda',\Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}; q, p_1, \dots, p_{2n_1-1}) \partial_{\vec{p}}^{w_2} \dot{C}^{\Lambda'}(q) \partial_{\vec{p}}^{w_3} \mathcal{L}_{2n_2,l_2}^{\Lambda',\Lambda_0}(p_{2n_1}, \dots, p_{2n}) \left. \right| \\
& \leq \sum_{\substack{l_1+l_2=l \\ n_1+n_2=n+1 \\ w_1+w_2+w_3=w}} 4n_1n_2 c_{\{w_j\}} \int_{\Lambda}^{\Lambda_0} d\Lambda' \Lambda'^{D-2n_1-|w_1|+4-2n_2-|w_2|} K^{(4n_1+8l_1-3+2n_2+4l_2-4)(|w_1|+D'-D+|w_2|+1)} \\
& \quad \times K^{D'(n_1+2l_1)^3} \sqrt{|v_1|! |v_2|!} \frac{\sqrt{(|w_1|+D'-D)!}}{|x|^{D'-D}} \sum_{\mu=0}^{d(2,n_1,l_1,w_1,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda'}\right)^{\mu} \\
& \quad \times \sum_{\lambda_1=0}^{2l_1+n_1} \frac{\log^{\lambda_1}(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m}))}{2^{\lambda_1} \lambda_1!} \frac{2}{\Lambda'^3} |\partial^{w_3} e^{-\frac{q^2+m^2}{\Lambda'^2}}| \sqrt{|w_2|!} (n_2+l_2-2)! \sum_{\lambda_2=0}^{l_2} \frac{\log^{\lambda_2}(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m}))}{2^{\lambda_2} \lambda_2!} \\
& \hspace{15cm} (4.78)
\end{aligned}$$

We use the inequality  $2l_1 + n_1 \leq 2l + n - 1$ , which holds due to the fact that the CAG without insertion vanishes unless  $n_2 + 2l_2 \geq 2$ , to obtain

$$\begin{aligned}
(4.78) & \leq K^{D'(2l+n-1)^3} K^{(4n+8l-7)(|w_1|+|w_2|+D'-D+1)} \sum_{\substack{l_1+l_2=l \\ n_1+n_2=n+1 \\ w_1+w_2+w_3=w}} 4n_1n_2 c_{\{w_j\}} (n_2+l_2-2)! \sqrt{|v_1|! |v_2|!} \\
& \quad \times \frac{\sqrt{(|w_1|+D'-D)! |w_2|!}}{|x|^{D'-D}} \int_{\Lambda}^{\Lambda_0} d\Lambda' \Lambda'^{D-2n-|w_1|-|w_2|-1} 2|\partial^{w_3} e^{-\frac{q^2+m^2}{\Lambda'^2}}| \sum_{\mu=0}^{d(2,n_1,l_1,w_1,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda'}\right)^{\mu} \\
& \quad \times \sum_{\lambda_1=0}^{2l_1+n_1} \sum_{\lambda_2=0}^{l_2} \frac{\log^{\lambda_1+\lambda_2}(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m}))}{2^{\lambda_1+\lambda_2} (\lambda_1+\lambda_2)!} \frac{(\lambda_1+\lambda_2)!}{\lambda_1! \lambda_2!} . \\
& \hspace{15cm} (4.79)
\end{aligned}$$

Then, using the bound [4, 20]

$$\left| \partial^w e^{-\frac{q^2+m^2}{\Lambda^2}} \right| \leq k \Lambda^{-|w|} \sqrt{|w|!} 2^{\frac{|w|}{2}} e^{-\frac{q^2}{2\Lambda^2}} e^{-\frac{m^2}{\Lambda^2}} , \quad k = 1.086 \dots \quad (4.80)$$

and the identity  $\lambda_1 + \lambda_2 \leq 2l_1 + n_1 + l_2 \leq 2l + n$  (since  $n_1 + n_2 = n + 1$  and the right side of the above bound is  $= 0$  for  $n_2 = 0$ ), we find the bound

$$\begin{aligned} \text{eq. (4.78)} &\leq \sum_{\substack{l_1+l_2=l, \\ n_1+n_2=n+1}} (n_2 + l_2)! 4n_1(2l + n) 2^{2l+n} K^{(4n+8l-7)(|w|+D'-D+1)} K^{D'(n+2l-1)^3} \\ &\times \sum_{w_i} c_{\{w_i\}} 2^{\frac{1}{2}|w_3|} k \frac{\sqrt{(|w_1| + D' - D)!}}{|x|^{D'-D}} \sqrt{|w_3|! |w_2|! |v_1|! |v_2|!} \\ &\times \sum_{\mu=0}^{d(2,n_1,l_1,w_1,D')} \frac{1}{\sqrt{\mu!}} \int_{\Lambda}^{\Lambda_0} d\Lambda' \Lambda'^{D-2n-|w|-1} \left(\frac{|\vec{p}|}{\Lambda}\right)^{\mu} \sum_{\lambda=0}^{2l+n} \frac{\log^{\lambda}(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m}))}{2^{\lambda} \lambda!}. \end{aligned} \quad (4.81)$$

Now the  $\Lambda'$  integral can again be estimated using the inequality (4.74). Noting that also

$$d(2, n_1, l_1, w_1, D') \leq d(2, n, l, w, D') \quad (4.82)$$

which holds as a result of the inequality  $2(n_1 + l_1) \leq 2(n + l) - 2$  (using  $n_1 + n_2 = n + 1$ , and  $n_2 + l_2 \geq 2$ ), and using also  $\sum c_{\{w_i\}} 2^{|w_3|/2} = (2 + \sqrt{2})^{|w|}$ , one obtains the bound claimed in (4.69) multiplied by  $1/8 \mathcal{A}$ , which was defined in eq.(4.76), provided that  $K$  is chosen large enough that

$$K^{-D'(n+2l)(n+2l-1)+(4n+8l-7)-(|w|+D'-D)} 5k \sum_{\substack{l_1+l_2=l \\ n_1+n_2=n+1}} (n_2 + l_2)! 4n_1(2l + n) 2^{2l+n} (2 + \sqrt{2})^{|w|} \leq \frac{1}{8} \quad (4.83)$$

To see that such a  $K$  can always be found, it is helpful to note the inequality  $(n + 2l)(n + 2l - 1) \geq 4n + 8l - 6$ .

**Third term on the r.h.s. of the flow equation (source term):** Note that this term is a momentum integral over the CAG's with one insertion, which have been bounded already, see (4.66). This integral obeys the following bound:

**Lemma 1.** *Let  $n_1 + n_2 = n + 1$  and  $l = l_1 + l_2$ . Then we have for any  $D \leq D'$*

$$\begin{aligned} &\left| \partial_{\vec{p}}^w \int_k \mathcal{L}_{2n_1, l_1}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}(x); k, p_1, \dots, p_{2n_1-1}) \dot{C}^{\Lambda}(k) \mathcal{L}_{2n_2, l_2}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_2}(0); -k, p_{2n_1}, \dots, p_{2n}) \right| \\ &\leq M K_0^{(4n+8l-4)(|w|+D'-D)} K_0^{D'(n+2l)^3} \sqrt{|v_1|! |v_2|!} \Lambda^{D-2n-|w|-1} e^{-m^2/\Lambda^2} \\ &\times \frac{\sqrt{(|w| + D' - D)!}}{|x|^{D'-D}} \sum_{\mu=0}^{d(N=2,n,l,w,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^{\mu} \sum_{\lambda=0}^{2l+n-1} \frac{\log^{\lambda}(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}) + \sqrt{\lambda!})}{2^{\lambda} \lambda!} \end{aligned} \quad (4.84)$$

where  $M = 5^{|w|+D'-D} 2^{2l+n+1} 2^{2d}(2l + n + 1) d 2^{[|w|+(D'-D)]/2}$  and where  $K_0$  is the constant (called  $K$  there) appearing in our bound on the CAG's with one insertion, see (4.66).



*Proof.* Note that we are bounding the same integral as in lemma 3.2 in [4]. Comparing both estimates one finds that in our bound we have effectively traded powers of  $\Lambda$  for inverse powers of  $|x|$ . This is achieved as follows: Using the translation covariance property of the CAG's [see discussion preceding eq.(3.33)], pulling momentum derivatives into the integral and performing partial integrations<sup>11</sup> we obtain (cf. [17] A.2 p.275)

$$\begin{aligned} & \left| \partial_{\vec{p}}^w \int_k \mathcal{L}_{2n_1, l_1}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}(x); k, p_1, \dots, p_{2n_1-1}) \dot{C}^\Lambda(k) \mathcal{L}_{2n_2, l_2}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_2}(0); -k, p_{2n_1}, \dots, p_{2n}) \right| \\ & \leq \left| \int_k \sum_{w_1+w_2+w_3=w} c_{\{w_i\}} e^{ikx} e^{ix(p_1+\dots+p_{2n_1-1})} \right. \\ & \quad \left. \partial_k^{w_3} \left( \partial_{\vec{p}}^{w_1} \mathcal{L}_{2n_1, l_1}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}(0); k, p_1, \dots, p_{2n_1-1}) \dot{C}^\Lambda(k) \partial_{\vec{p}}^{w_2} \mathcal{L}_{2n_2, l_2}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_2}(0); -k, p_{2n_1}, \dots, p_{2n}) \right) \right| \end{aligned} \quad (4.85)$$

Denote by  $\|x\| = \max_{\alpha \in \{1, \dots, 4\}} |x_\alpha|$  the maximal component of  $x$ . We can multiply both sides of (4.85) by  $(2\|x\|)^{D'-D}$ , pull this factor into the integral, and we use the elementary fact that  $\|x\|^{D'-D} e^{ikx}$  can be written as a  $(D-D')$ -fold  $k$ -derivative on  $e^{ikx}$  with respect to a suitable component of the 4-vector  $k$ . That  $k$ -derivative may then be moved onto the term in the last line of (4.85) via a partial integration. The resulting expression can now be bounded just as in lemma 3.2 in [4] by substituting the bounds for the CAG's with one insertion given in that paper. After some bookkeeping we obtain the following bound for the right side of (4.85)

$$\begin{aligned} (4.85) \times (2\|x\|)^{D'-D} & \leq M_0 K_0^{(4n+8l-4)(|w|+D'-D)} K_0^{D'(n+2l)^3} \sqrt{(|w|+D'-D)!|v_1|!|v_2|!} \\ & \times \Lambda^{D-2n-|w|-5} e^{-m^2/\Lambda^2} \int_k e^{-k^2/2\Lambda^2} \sum_{\mu=0}^{d_1+d_2} \frac{1}{\sqrt{\mu}!} \left( \frac{|\vec{p}|+|k|}{\Lambda} \right)^\mu \sum_{\lambda=0}^{2l+n-1} \frac{\log^\lambda(\sup(\frac{|\vec{p}|+|k|}{\kappa}, \frac{\kappa}{m})}{2^\lambda \lambda!} \end{aligned} \quad (4.86)$$

where  $M_0 = 5^{|w|+D'-D} 2^{2l+n} 2^{d_1+d_2} (2l+n)(d_1+d_2) 2^{\lfloor |w|+D'-D \rfloor / 2}$  and where we used the shorthand  $d_i = 2[A_i](n_i + l_i) + \sup([A_i] + 1 - 2n_i - |w_i|, 0)$ . We would now like to replace  $d_1 + d_2$  by  $d(2, n, l, w, D')$  in the summation index. Here it is crucial that  $d$  satisfies the property

$$d(2, n, l, w, D') \geq d_1 + d_2 \quad (4.87)$$

with  $d_i = 2[A_i](n_i + l_i) + \sup([A_i] + 1 - 2n_i - |w_i|, 0)$  and where  $n_1 + n_2 = n + 1, l_1 + l_2 = l, w_1 + w_2 = w$  and  $[A_1] + [A_2] = D'$ . Substituting the definition of  $d$  given in the statement of theorem 3 we indeed find

$$d(2, n, l, w, D') = 2D'(n + l + 2) + \sup(D' + 1 - 2n - |w|, 0) \geq 2D'(n + l + 1) + 2D' \geq d_1 + d_2 \quad (4.88)$$

---

<sup>11</sup>The integrand decays sufficiently rapidly for large  $k$  to justify partial integrations, due to the exponential damping factor in  $\dot{C}^\Lambda(k)$ . According to the bounds stated in eq.(4.66) the CAG's with one insertion only grow polynomially in  $k$ .

for arbitrary  $w$ . The momentum integral can then be bounded by an application of the inequality (4.73).

$$\begin{aligned} & \sum_{\mu=0}^d \frac{1}{\sqrt{\mu!}} \int_{k/\Lambda} e^{-|k|^2/2\Lambda^2} \left( \frac{|\vec{p}| + |k|}{\Lambda} \right)^\mu \sum_{\lambda=0}^{2l+n-1} \frac{\log^\lambda(\sup(\frac{|\vec{p}|+|k|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!} \\ & \leq 2^d \sum_{\mu=0}^d \frac{1}{\sqrt{\mu!}} \left( \frac{|\vec{p}|}{\Lambda} \right)^\mu \sum_{\lambda=0}^{2l+n-1} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})) + \sqrt{\lambda!}}{2^\lambda \lambda!} \end{aligned} \quad (4.89)$$

Finally, we divide again by  $(2\|x\|)^{D'-D}$  and note the elementary inequality

$$|x| = \sqrt{x_1^2 + \dots + x_4^2} \leq 2\|x\| \quad (4.90)$$

to obtain the bound claimed in the lemma.  $\square$

Now, to obtain a bound for the third term in the FE we have to integrate the bound derived in lemma 1 over  $\Lambda'$  between  $\Lambda_0$  and  $\Lambda$ . Using the identity (4.74) for the  $\Lambda'$  integral, we obtain the bound

$$\begin{aligned} & \left| \int_{\Lambda} d\Lambda' \partial_{\vec{p}}^w \int_k \mathcal{L}_{2n_1, l_1}^{\Lambda', \Lambda_0}(\mathcal{O}_{A_1}; k, p_1, \dots, p_{2n_1-1}) \dot{C}^{\Lambda'}(k) \mathcal{L}_{2n_2, l_2}^{\Lambda', \Lambda_0}(\mathcal{O}_{A_2}; -k, p_{2n_1}, \dots, p_{2n}) \right| \\ & \leq M K_0^{(4n+8l-4)(|w|+D'-D)} K_0^{D'(n+2l)^3} \frac{\sqrt{(|w|+D'-D)!}}{|x|^{D'-D}} \sqrt{|v_1|! |v_2|!} \\ & \times \int_{\Lambda} d\Lambda' \Lambda'^{-1+D-2n-|w|} e^{-m^2/\Lambda'^2} \sum_{\mu=0}^{d(2,n,l,w,D')} \frac{1}{\sqrt{\mu!}} \left( \frac{|\vec{p}|}{\Lambda'} \right)^\mu \sum_{\hat{a}=0}^{2l+n-1} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m})) + \sqrt{\lambda!}}{2^\lambda \lambda!} \\ & \leq 5 M K_0^{(4n+8l-4)(|w|+D'-D)} K_0^{D'(n+2l)^3} \frac{\sqrt{(|w|+D'-D)!}}{|x|^{D'-D}} \sqrt{|v_1|! |v_2|!} \\ & \times \Lambda^{D-2n-|w|} \sum_{\mu=0}^{d(2,n,l,w,D')} \frac{1}{\sqrt{\mu!}} \left( \frac{|\vec{p}|}{\Lambda} \right)^\mu \sum_{\lambda=0}^{2l+n-1} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!} \end{aligned} \quad (4.91)$$

where  $K_0$  is the constant from lemma 1. Recall from the FE, eq.(4.70), that this term is to be multiplied by  $4n_1n_2$ , and we also have to apply the operator  $\mathbb{S}$  and sum over the configurations  $n_1 + n_2 = n + 1$  and  $l_1 + l_2 = l$ . In total we find that the inductive bound is reproduced under the condition that  $K$  satisfies

$$20(l+1)(n+1)^3 M K_0^{(4n+8l-4)(|w|+D'-D)} K_0^{D'(n+2l)^3} \leq \frac{1}{8} K^{(4n+8l-3)(|w|+D'-D)} K^{D'(n+2l)^3}. \quad (4.92)$$

Here it is useful to note that both  $(2l + n)$  and  $(4n + 8l - 3)$  are always positive (since  $n + l \geq 1$ ), which helps one to see that the inequality can be satisfied by making  $K$  large enough.

**(B) Relevant terms ( $2n + |w| \leq D$ ) at  $\vec{p} = \vec{0}$ :**

As the boundary conditions for the relevant terms are given at zero momentum, we will first derive bounds for  $\vec{p} = \vec{0}$  and then proceed to arbitrary momentum with the help of the Taylor expansion formula

$$\partial_{\vec{p}}^w f_{2n}(\vec{p}) = \sum_{|\tilde{w}| \leq D-2n-|w|} \frac{\vec{p}^{\tilde{w}}}{\tilde{w}!} [\partial_{\vec{p}}^{\tilde{w}+w} f_{2n}](0) + \sum_{|\tilde{w}|=D+1-2n-|w|} \vec{p}^{\tilde{w}} \int_0^1 d\tau \frac{(1-\tau)^{|\tilde{w}|-1}}{(|\tilde{w}|-1)!} [\partial_{\vec{p}}^{\tilde{w}+w} f_{2n}](\tau \vec{p}) . \quad (4.93)$$

**First term on the r.h.s. of the FE:** In view of eq.(4.93), let us consider the first term on the r.h.s. of the FE with momentum derivatives  $\partial_{\vec{p}}^{\tilde{w}+w}$  and with  $2n + |\tilde{w} + w| \leq D$  at zero momentum. Integrating over  $\Lambda'$  from 0 to  $\Lambda$  we find

$$\begin{aligned} & \left| \int_0^\Lambda d\Lambda' \binom{2n+2}{2} \int d^4k \dot{C}^{\Lambda'}(k) \partial_{\vec{p}}^{w+\tilde{w}} \mathcal{L}_{D,2n+2,l-1}^{\Lambda',\Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}; k, -k, 0, \dots, 0) \right| \\ & \leq \binom{2n+2}{2} K^{(4n+8l-7)(|\tilde{w}+w|+D'-D)} K^{D'(n+2l-1)^3} \sqrt{|v_1|!|v_2|!} \int_0^\Lambda d\Lambda' \Lambda'^{D-(2n+2)-|\tilde{w}+w|} \frac{2}{\Lambda'^3} \\ & \times \frac{\sqrt{(|w+\tilde{w}|+D'-D)!}}{|x|^{D'-D}} \int d^4k e^{-\frac{k^2+m^2}{\Lambda'^2}} \sum_{\mu=0}^{d(2,n,l,\tilde{w}+w,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|k|}{\Lambda'}\right)^\mu \sum_{\lambda=0}^{2l+n-1} \frac{\log^\lambda(\sup(\frac{|k|}{\kappa'}, \frac{\kappa'}{m}))}{2^\lambda \lambda!} \\ & \leq \binom{2n+2}{2} K^{(4n+8l-7)(|\tilde{w}+w|+D'-D)} K^{D'(n+2l-1)^3} \frac{\sqrt{|v_1|!|v_2|!} (|\tilde{w}+w|+D'-D)!}{|x|^{D'-D}} \sum_{\mu=0}^{d(2,n,l,\tilde{w}+w,D')} \\ & \times \sum_{\lambda=0}^{2l+n-1} \frac{2}{2^\lambda \lambda!} \int_0^\Lambda d\Lambda' \Lambda'^{D-(2n+1)-|\tilde{w}+w|} e^{-\frac{m^2}{\Lambda'^2}} \int d^4(k/\Lambda') \frac{1}{\sqrt{\mu!}} \left(\frac{|k|}{\Lambda'}\right)^\mu \log^\lambda(\sup(\frac{|k|}{\kappa'}, \frac{\kappa'}{m})) e^{-\frac{k^2}{\Lambda'^2}} \end{aligned} \quad (4.94)$$

The momentum integral in the last line can be estimated by (cf. eq.(76) in [4])

$$2^{\frac{\mu}{2}} \frac{1}{\sqrt{\mu!}} \left(\frac{\mu}{2}\right)! [\log^\lambda(\frac{\kappa'}{m}) + (\lambda!)^{1/2}] . \quad (4.95)$$

and for the subsequent sum over  $\mu$  we can use the bound (eq.(88) in [4])

$$\sum_{\mu=0}^d 2^{\frac{\mu}{2}} \frac{1}{\sqrt{\mu!}} \left(\frac{\mu}{2}\right)! \leq 2 d^{3/2} . \quad (4.96)$$

For the  $\Lambda'$  integral we make use of the inequality (eq.(89) in [4])

$$\begin{aligned} & \int_0^\Lambda d\Lambda' \Lambda'^{D-(2n+1)-|\tilde{w}+w|} \log^\lambda\left(\frac{\kappa'}{m}\right) e^{-\frac{m^2}{\Lambda'^2}} \\ & \leq \Lambda^{D-2n-|\tilde{w}+w|} \begin{cases} \log^\lambda\left(\frac{\kappa'}{m}\right) & \text{if } D-2n-|\tilde{w}+w| > 0, \\ 2(\lambda+1)^{-1} \log^{\lambda+1}\left(\frac{\kappa'}{m}\right) & \text{if } D-2n-|\tilde{w}+w| = 0. \end{cases} \end{aligned} \quad (4.97)$$

Using these bounds to estimate the r.h.s. of (4.94) shows that this contribution satisfies the inductive bound, formula (4.69), multiplied by  $1/8 \mathcal{A}$ , defined in eq.(4.76), provided that  $K$  is chosen such that

$$12 \binom{2n+2}{2} d(2, n, l, \tilde{w}+w, D')^{3/2} K^{-D'(n+2l)(n+2l-1)-(|w+\tilde{w}|+D'-D)} \leq 1/8. \quad (4.98)$$

It can be seen that this is indeed possible to find such a  $K$ . This is easy to see for large values of  $n+l$ . To see that it is also true for small  $n+l$ , it is useful to recall that for the first term on the r.h.s. of the FE, (4.70), we can assume  $l \geq 1$ , so  $K$  will always have a negative exponent.

**Second term on the r.h.s of the FE:** Inserting the induction hypothesis for the CAG's with one insertion and the known bounds for the CAG's without insertion, formula (4.63), (4.64) and (4.65), yields

$$\begin{aligned} & \left| \int_0^\Lambda d\Lambda' \sum_{\substack{l_1+l_2=l \\ n_1+n_2=n+1 \\ w_1+w_2+w_3=w+\tilde{w}}} 4n_1n_2 c_{\{w_j\}} \right. \\ & \quad \times \left. \partial_{\vec{p}}^{w_1} \mathcal{L}_{D,2n_1,l_1}^{\Lambda',\Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}; q, p_1, \dots, p_{2n_1-1}) \partial_{\vec{p}}^{w_3} \dot{C}^{\Lambda'}(q) \partial_{\vec{p}}^{w_2} \mathcal{L}_{2n_2,l_2}^{\Lambda',\Lambda_0}(p_{2n_1}, \dots, p_{2n}) \right|_{\vec{p}=\vec{0}} \\ & \leq \sum_{\substack{l_1+l_2=l, \\ w_1+w_2+w_3=w+\tilde{w}, \\ n_1+n_2=n+1}} 4n_1n_2 \int_0^\Lambda d\Lambda' \Lambda'^{D+4-2n-2-|w_1|-|w_2|} K^{(4n_1+8l_1-3+2n_2+4l_2-4)(|w_1|+D'-D+|w_2|+1)} \\ & \quad \times K^{D'(n_1+2l_1)^3} \sqrt{|v_1|! |v_2|!} \frac{\sqrt{(|w_1|+D'-D)!}}{|x|^{D'-D}} c_{\{w_i\}} \sum_{\lambda_1=0}^{2l_1+n_1} \frac{\log^{\lambda_1}\left(\frac{\kappa'}{m}\right)}{2^{\lambda_1} \lambda_1!} \\ & \quad \times \frac{2}{\Lambda'^3} \left| \partial^{w_3} e^{-\frac{q^2+m^2}{\Lambda'^2}} \right|_{q=0} \sqrt{|w_2|!} (n_2+l_2-2)! \sum_{\lambda_2=0}^{l_2} \frac{\log^{\lambda_2}\left(\frac{\kappa'}{m}\right)}{2^{\lambda_2} \lambda_2!} \end{aligned} \quad (4.99)$$

We again use the inequality (4.80) and proceed as in (4.97) to estimate the  $\Lambda'$  integral. As a result we arrive at the bound (4.69) multiplied by  $1/8 \mathcal{A}$ , under the condition that  $K$

satisfies the lower bound

$$6 K^{-D'(n+2l)(n+2l-1)+(4n+8l-7)-(|w+\tilde{w}|+D'-D)} 5k \times \sum_{\substack{l_1+l_2=l, \\ n_1+n_2=n+1}} (n_2+l_2)! 4 n_1 (2l+n) 2^{2l+n} (2+\sqrt{2})^{|w+\tilde{w}|} \leq \frac{1}{8}. \quad (4.100)$$

The inequality  $(n+2l)(n+2l-1) \geq (4n+8l-6)$  ensures that  $K$  always appears with a negative exponent on the right side, which is helpful in order to see that such a  $K$  can be found.

**Third term on the r.h.s. of the FE :** Using lemma 1, we find

$$\left| \int_0^\Lambda d\Lambda' \partial_{\vec{p}}^{w+\tilde{w}} \int_k \mathcal{L}_{2n_1, l_1}^{\Lambda', \Lambda_0}(\mathcal{O}_A; k, 0, \dots, 0) \dot{C}^{\Lambda'}(k) \mathcal{L}_{2n_2, l_2}^{\Lambda', \Lambda_0}(\mathcal{O}_B; -k, 0, \dots, 0) \right| \leq M K_0^{(4n+8l-4)(|w|+|\tilde{w}|+D'-D)} K_0^{D'(n+2l)^3} \sqrt{|v_1|! |v_2|!} \frac{\sqrt{(|w|+|\tilde{w}|+D'-D)!}}{|x|^{D'-D}} \times \int_0^\Lambda d\Lambda' \Lambda'^{D-2n-|w|-|\tilde{w}|-1} e^{-m^2/\Lambda'^2} \sum_{\lambda=0}^{2l+n-1} \frac{\log^\lambda(\frac{\kappa'}{m}) + \sqrt{\lambda!}}{2^\lambda \lambda!}. \quad (4.101)$$

For the integral over  $\Lambda'$  we use the inequality [4]

$$\int_0^\Lambda d\Lambda' \Lambda'^{D-2n-|w|-|\tilde{w}|-1} e^{-m^2/\Lambda'^2} \sum_{\lambda=0}^{2l+n-1} \frac{\log^\lambda(\frac{\kappa'}{m}) + \sqrt{\lambda!}}{2^\lambda \lambda!} \leq \Lambda^{D-2n-|w|-|\tilde{w}|} \sum_{\lambda=0}^{2l+n-1} \frac{1}{2^\lambda \lambda!} \begin{cases} \frac{1}{\lambda+1} \log^{\lambda+1}(\frac{\kappa'}{m}) + \sqrt{\lambda!} \log(\frac{\kappa'}{m}) + 1 & \text{if } D-2n-|w| = |\tilde{w}|, \\ \frac{\log^\lambda(\frac{\kappa'}{m}) + \sqrt{\lambda!}}{D-2n-|w|-|\tilde{w}|} & \text{if } D-2n-|w| > |\tilde{w}| \end{cases} \leq 6(l+1) \Lambda^{D-2n-|w|-|\tilde{w}|} \sum_{\lambda=0}^{2l+n} \frac{\log^\lambda(\frac{\kappa'}{m})}{2^\lambda \lambda!}. \quad (4.102)$$

Recalling that we have to multiply (4.101) by  $4n_1n_2$  and that we also have to apply the symmetrization operator  $\mathbb{S}$  and sum over the indices  $n_i$  and  $l_i$ , we reproduce the inductive bound under the condition

$$4 \cdot 6(l+1)(2l+n) (n+1)^3 M K_0^{D'(n+2l)^3} K_0^{(4n+8l-4)(|w|+|\tilde{w}|+D'-D)} \leq \frac{1}{8} K^{D'(n+2l)^3} K^{(4n+8l-3)(|w|+|\tilde{w}|+D'-D)}, \quad (4.103)$$

on  $K$ . Again, this condition can be satisfied by a sufficiently large  $K$ .

**(C) Relevant terms ( $2n + |w| \leq D$ ) at arbitrary momentum:**

In order to proceed to non-zero momentum we now make use of the Taylor series.

$$\begin{aligned}
|\partial_{\vec{p}}^w \mathcal{L}_{D,2n,l}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}; \vec{p})| &= \left| \sum_{|\tilde{w}| \leq D-2n-|w|} \frac{\vec{p}^{\tilde{w}}}{\tilde{w}!} \partial_{\vec{p}}^{\tilde{w}+w} \mathcal{L}_{D,2n,l}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}; \vec{0}) \right. \\
&\quad \left. + \sum_{|\tilde{w}|=D+1-2n-|w|} \vec{p}^{\tilde{w}} \int_0^1 \frac{(1-\tau)^{|\tilde{w}|-1}}{(|\tilde{w}|-1)!} \partial_{\tau \vec{p}}^{\tilde{w}+w} \mathcal{L}_{D,2n,l}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}; \tau \vec{p}) \, d\tau \right|
\end{aligned} \tag{4.104}$$

On the right hand side we can now use the bounds previously derived in **(A)** and **(B)**.

$$\begin{aligned}
&|\partial_{\vec{p}}^w \mathcal{L}_{D,2n,l}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}; \vec{p})| \\
&\leq \sum_{|\tilde{w}| \leq D-2n-|w|} \Lambda^{D-2n-|w|} \left( \frac{|\vec{p}|}{\Lambda} \right)^{|\tilde{w}|} \frac{\sqrt{|v_1|!|v_2|!}(|w|+|\tilde{w}|+D'-D)!}{\tilde{w}!|x|^{D'-D}} \sum_{\lambda=0}^{2l+n} \frac{\log^\lambda(\frac{\kappa}{m})}{2^\lambda \lambda!} \\
&\quad \times \left[ 1/4 \mathcal{A}(D', D, n, l, w + \tilde{w}) K^{(4n+8l-3)(|w|+|\tilde{w}|+D'-D)+D'(2l+n)^3} \right. \\
&\quad \left. + 24M(2l+n)(l+1)(n+1)^3 K_0^{(4n+8l-4)(|w|+|\tilde{w}|+D'-D)+D'(2l+n)^3} \right] \\
&+ \sum_{|\tilde{w}|=D+1-2n-|w|} \Lambda^{D-2n-|w|} \left( \frac{|\vec{p}|}{\Lambda} \right)^{|\tilde{w}|} \frac{\sqrt{|v_1|!|v_2|!}(|w|+|\tilde{w}|+D'-D)!}{|\tilde{w}||x|^{D'-D}} |\tilde{w}| \\
&\quad \times \int_0^1 d\tau (1-\tau)^{|\tilde{w}|-1} \sum_{\mu=0}^{d(2,n,l,w+\tilde{w},D')} \frac{1}{\sqrt{\mu}!} \left( \frac{\tau|\vec{p}|}{\Lambda} \right)^\mu \sum_{\lambda=0}^{2l+n} \frac{\log^\lambda(\sup(\frac{\tau|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!} \\
&\quad \times \left[ 1/4 \mathcal{A}(D', D, n, l, w + \tilde{w}) K^{(4n+8l-3)(|w|+|\tilde{w}|+D'-D)+D'(2l+n)^3} \right. \\
&\quad \left. + 20M(l+1)(n+1)^3 K_0^{(4n+8l-4)(|w|+|\tilde{w}|+D'-D)+D'(2l+n)^3} \right]
\end{aligned} \tag{4.105}$$

Here  $K_0$  is the constant (called  $K$  there) from the bound on the CAG's with one insertion, (4.68), and  $M$  is the constant defined in the statement of lemma 1. To obtain the r.h.s. of the inequality above, we used the fact that the first and second term on the r.h.s. of the FE satisfy the bound (4.69) multiplied by  $1/8 \mathcal{A}$  in the cases **(A)** and **(B)**. Hence the expressions include the factor  $\mathcal{A}$ . For the contribution from the third term in the FE we also used the bounds derived above for the cases **(A)** and **(B)**, but expressed in terms of the constant  $K_0$  instead of  $K$ , see the inequalities (4.91) and (4.101) and the discussion following them.

To simplify this expression, we make use of:

$$\sqrt{(|w| + |\tilde{w}| + D' - D)!} \leq \sqrt{|\tilde{w}||w| + D' - D)!} 2^{(|w|+|\tilde{w}|+D'-D)/2} \tag{4.106}$$

$$\sum_{|\tilde{w}|=j} \frac{|\tilde{w}|!}{\tilde{w}!} = (8n)^j \tag{4.107}$$

$$\frac{1}{\sqrt{|\tilde{w}|!}} \sum_{\mu=0}^d \frac{1}{\sqrt{\mu!}} \left( \frac{|\vec{p}|}{\Lambda} \right)^{\mu+|\tilde{w}|} \leq \sum_{\mu=0}^{d+\tilde{w}} \frac{1}{\sqrt{\mu!}} \left( \frac{|\vec{p}|}{\Lambda} \right)^{\mu} 2^{\mu/2} \quad (4.108)$$

$$d(2, n, l, \tilde{w} + w, D') + |\tilde{w}| \leq d(2, n, l, w, D') \quad \text{for } |\tilde{w}| \leq D' - 2n - |w| + 1 \quad (4.109)$$

We then obtain the bound

$$\begin{aligned} & |\partial_{\vec{p}}^w \mathcal{L}_{D,2n,l}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}; \vec{p})| \\ & \leq \Lambda^{D-2n-|w|} \frac{\sqrt{|v_1|!|v_2|!}(|w|+D'-D)!}{|x|^{D'-D}} \sum_{\mu=0}^{d(2,n,l,w,D')} \frac{1}{\sqrt{\mu!}} \left( \frac{|\vec{p}|}{\Lambda} \right)^{\mu} \sum_{\lambda=0}^{2l+n} \frac{\log^{\lambda}(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^{\lambda} \lambda!} \\ & \quad \times \left( \sum_{|\tilde{w}| \leq D-2n-|w|} \left[ 1/4 \mathcal{A}(D', D, n, l, w + \tilde{w}) K^{(4n+8l-3)(|w|+|\tilde{w}|+D'-D)+D'(2l+n)^3} \right. \right. \\ & \quad \left. \left. + 24M(2l+n)(l+1)(n+1)^3 K_0^{(4n+8l-4)(|w|+|\tilde{w}|+D'-D)+D'(2l+n)^3} \right] 2^{(|w|+|\tilde{w}|+D'-D)/2} \frac{(8n)^{|\tilde{w}|}}{\sqrt{|\tilde{w}|!}} \right) \\ & \quad \times \left( \sum_{|\tilde{w}| \leq D+1-2n-|w|} \left[ 1/4 \mathcal{A}(D', D, n, l, w + \tilde{w}) K^{(4n+8l-3)(|w|+|\tilde{w}|+D'-D)+D'(2l+n)^3} \right. \right. \\ & \quad \left. \left. + 20M(l+1)(n+1)^3 K_0^{(4n+8l-4)(|w|+|\tilde{w}|+D'-D)+D'(2l+n)^3} \right] 2^{(|w|+|\tilde{w}|+D'-D)/2} |\tilde{w}| 2^{d/2} \right) \end{aligned} \quad (4.110)$$

It follows, using also (4.107), that this contribution satisfies the bound claimed in the theorem, (4.69), provided that  $K$  is chosen sufficiently large such that the following two conditions are satisfied:

$$\sum_{j \leq D+1-2n-|w|} (8n)^j j 2^{\frac{d(2,n,l,w,D')}{2}} 2^{\frac{j+|w|+D'-D}{2}} K^{(4n+8l-3)j} \mathcal{A}(D', D, n, l, |w|+j) \leq 1 \quad (4.111)$$

$$\begin{aligned} & \sum_{j \leq D+1-2n-|w|} 48M(2l+n)(l+1)(n+1)^3 K_0^{(4n+8l-4)(|w|+j+D'-D)+D'(2l+n)^3} \\ & \quad \times (8n)^j j 2^{\frac{d(2,n,l,w,D')}{2}} 2^{\frac{j+|w|+D'-D}{2}} \leq K^{(4n+8l-3)(|w|+D'-D)+D'(2l+n)^3} \end{aligned} \quad (4.112)$$

It can be seen that it is indeed possible to choose  $K$  such that the conditions (4.111) and (4.112) are satisfied, which finishes the proof of theorem 3.  $\square$

The following corollary is a consequence of thm. 3.

**Corollary 1.** *For  $\Lambda \leq m$  there exists a constant  $K > 0$  such that*

$$\begin{aligned} & |\partial_{\vec{p}}^w \mathcal{L}_{D,2n,l}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_1}(x) \otimes \mathcal{O}_{A_2}(0); \vec{p})| \leq m^{D-2n-|w|} K^{(4n+8l-3)(|w|+D'-D)} K^{D'(n+2l)^3} \sqrt{|v_1|!|v_2|!} \\ & \quad \times \frac{\sqrt{(|w|+D'-D)!(2n+|w|-D)_+!}}{|x|^{D'-D}} \sup \left( 1, \frac{|\vec{p}|}{m} \right)^{d(2,n,l,w,D')} \sum_{\lambda=0}^{2l+n} \frac{\log^{\lambda}(\frac{|\vec{p}|}{m})}{2^{\lambda} \lambda!} \end{aligned} \quad (4.113)$$

with  $d(N, n, l, w, D') := 2D'(n + l + 2(N - 1)) + \sup(D' + 1 - 2n - |w|, 0)$  and  $A_i \equiv \{n_i, v_i\}$ .

**Remark:** This bound implies convergence of the Taylor expansion of  $\partial_{\vec{p}}^w \mathcal{L}_{D, 2n, l}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}(x) \otimes \mathcal{O}_{A_2}(0))$  with respect to  $x$  in a neighborhood of  $x \neq 0$ , for any degree of regularization  $D \leq [A_1] + [A_2]$ , and uniformly in  $\Lambda, \Lambda_0$ . This can be seen from the fact that the bound grows like

$$j! \left( \sup(|\vec{p}|/m, 1)^{2(n+l+2)+1} \tilde{K}/|x| \right)^j \quad (4.114)$$

if we take  $j$  derivatives with respect to  $x$  and apply the Lowenstein rule (3.47). Hence the CAG's with two insertions are real analytic in  $x$  for  $x \neq 0$ , and similarly for the OPE coefficients, related to them by defn. (1).

*Proof.* We can insert the bounds from theorem 3 into the FE (4.70) once more and integrate over  $\Lambda$  from 0 to  $m$ . Note that there is a damping factor  $\exp(-m^2/\Lambda^2)$  in each of the terms on the r.h.s. of the FE, so we can bound negative powers of  $\Lambda$  through the estimate

$$\begin{aligned} \int_0^m d\Lambda' \exp(-m^2/\Lambda'^2) \frac{\Lambda'^{D-2n-|w|-\mu-1}}{\sqrt{\mu}!} &\leq m^{D-2n-|w|-\mu} \frac{\sqrt{(2n+|w|+\mu-D)_+!}}{\sqrt{\mu}!} \\ &\leq 2^{(2n+|w|+\mu-D)_+/2} m^{D-2n-|w|-\mu} \sqrt{(2n+|w|-D)_+!} \end{aligned} \quad (4.115)$$

Choosing a somewhat larger constant  $K$  to accommodate for the additional powers of 2 we obtain the corollary.  $\square$

### 4.3 CAG's with three insertions

We next derive a bound on the CAG's  $L_{\mathbf{D}}^{\Lambda, \Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i})$  with three insertions and a set  $\mathbf{D} = (D_I \mid I \in \mathcal{I})$  of regulating parameters, as defined in sec. 3.3. Here  $\mathcal{I}$  always includes the set  $\{1, 2, 3\}$ , and it can include any number of two element subsets. We thus have the choices:

$$\mathbf{D} = \begin{cases} (D_{\{1,2,3\}}, D_{\{1,2\}}) & \text{or} \\ (D_{\{1,2,3\}}, D_{\{1,3\}}) & \text{or} \\ (D_{\{1,2,3\}}, D_{\{2,3\}}) & \text{or} \\ (D_{\{1,2,3\}}, D_{\{1,2\}}, D_{\{2,3\}}, D_{\{1,3\}}) & . \end{cases} \quad (4.116)$$

Since the first three are obviously related simply by a permutation of the insertions, and since the regulated CAG's have appropriate symmetry properties (from their FE and boundary conditions) we only need to consider one, say the first, case. A formula relating different choices of the set of regulating parameters is

$$L_{(D_{\{1,2,3\}}, D_{\{1,2\}}, D_{\{2,3\}}, D_{\{1,3\}})}^{\Lambda, \Lambda_0} = L_{(D_{\{1,2,3\}}, D_{\{1,2\}})}^{\Lambda, \Lambda_0} + L_{(D_{\{1,2,3\}}, D_{\{2,3\}})}^{\Lambda, \Lambda_0} + L_{(D_{\{1,2,3\}}, D_{\{1,3\}})}^{\Lambda, \Lambda_0} . \quad (4.117)$$

This formula can be proved by noting that the left and right sides satisfy the same FE and boundary conditions. As a consequence, the bounds that we will derive for the quantities on the right side will imply bounds on the quantity on the left side. Our bound is:



**Proposition 1.** Let  $\mathbf{D} = (D_{\{1,2,3\}}, D_{\{1,2\}})$ , with regularization parameters such that  $\delta_1 := [A_1] + [A_2] - D_{\{1,2\}} \geq 0$ ,  $\delta_2 := D_{\{1,2\}} + [A_3] - D_{\{1,2,3\}} \geq 0$  and  $D_{\{1,2,3\}} \leq D' = [A_1] + [A_2] + [A_3]$ . There exists a constant  $K > 0$  such that

$$\begin{aligned} |\mathcal{L}_{\mathbf{D}, 2n, l}^{\Lambda, \Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}(x_i); \vec{p})| &\leq K^{(4n+8l-3)(D'-D_{\{1,2,3\}})} K^{D'(n+2l)^3} \\ &\times \frac{\Lambda^{D_{\{1,2,3\}}-2n}}{|x_1 - x_2|^{\delta_1} \max(|x_1 - x_3|, |x_2 - x_3|)^{\delta_2}} \\ &\times \sqrt{|v_1|!|v_2|!|v_3|!(D' - D_{\{1,2,3\}})!} \sum_{\mu=0}^{d(3,n,l,0,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^\mu \sum_{\lambda=0}^{2l+n+1} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}), \frac{\kappa}{m})}{2^\lambda \lambda!} \end{aligned} \quad (4.118)$$

with  $d(N, n, l, w, D') := 2D'(n + l + 2(N - 1)) + \sup(D' + 1 - 2n - |w|, 0)$ , and  $A_i \equiv \{n_i, v_i\}$ .

**Remarks:** Here we can see explicitly how the regularization parameters allow us to remove (sub)divergences.

1. The divergence on the total diagonal is of order  $\delta_1 + \delta_2 = D' - D_{\{1,2,3\}}$ . Thus,  $D_{\{1,2,3\}}$  controls the degree of divergence on the total diagonal.
2. It follows by definition that as we increase  $D_{\{1,2\}}$ , we decrease  $\delta_1$  and increase  $\delta_2$ . We see that in the bound this cures the divergence on the partial diagonal  $x_1 = x_2$  without changing the degree of divergence on the total diagonal. Hence,  $D_{\{1,2\}}$  is indeed a regularization parameter associated to the partial diagonal  $x_1 = x_2$ .
3. Note also that the bound above is regular on the other partial diagonals,  $x_2 = x_3$  and  $x_1 = x_3$  (where  $x_1 \neq x_2$ ). In other words, the decomposition of the CAG in eq.(4.117) is essentially a decomposition into contributions with a different nesting of subdivergences.

The proposition is an obvious consequence of the following theorem:

**Theorem 4.** Let  $x_3 = 0$  and  $\mathbf{D} = (D_{\{1,2,3\}}, D_{\{1,2\}})$ , with regularization parameters such that  $\delta_1 := [A_1] + [A_2] - D_{\{1,2\}} \geq 0$ ,  $\delta_2 := D_{\{1,2\}} + [A_3] - D_{\{1,2,3\}} \geq 0$  and  $D_{\{1,2,3\}} \leq D' = [A_1] + [A_2] + [A_3]$ . There exists a constant  $K > 0$  such that

$$\begin{aligned} |\partial_{\vec{p}}^w \mathcal{L}_{\mathbf{D}, 2n, l}^{\Lambda, \Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}(x_i); \vec{p})| &\leq K^{(4n+8l-3)(|w|+D'-D_{\{1,2,3\}})} K^{D'(n+2l)^3} \\ &\times \frac{\Lambda^{D_{\{1,2,3\}}-2n-|w|}}{|x_1 - x_2|^{\delta_1} \max(|x_1|, |x_2|)^{\delta_2}} \\ &\times \sqrt{|v_1|!|v_2|!|v_3|!(|w| + D' - D_{\{1,2,3\}})!} \sum_{\mu=0}^{d(3,n,l,w,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^\mu \sum_{\lambda=0}^{2l+n+1} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}), \frac{\kappa}{m})}{2^\lambda \lambda!} \end{aligned} \quad (4.119)$$

with  $d(N, n, l, w, D') := 2D'(n + l + 2(N - 1)) + \sup(D' + 1 - 2n - |w|, 0)$ , and  $A_i \equiv \{n_i, v_i\}$ .

**Remark:** Note that in the case  $w = 0$  we can make use of the translation covariance properties of the CAG's, eq.(3.33), in order to remove the condition  $x_3 = 0$  and thus to obtain proposition 1. However, the translation operator  $\exp(ix_3(p_1 + \dots + p_{2n}))$  evidently does not commute with the momentum derivatives  $\partial_{\vec{p}}^w$ , which is the reason why we did not include any momentum derivatives in proposition 1.

*Proof of theorem 4.* The FE with additional momentum derivatives for the CAG's under consideration is in the case at hand [cf. (3.43) and recall that we have  $\mathcal{I} = (\{1, 2, 3\}, \{1, 2\})$  in the case at hand]

$$\begin{aligned}
& \partial_\Lambda \partial_{\vec{p}}^w \mathcal{L}_{\mathbf{D}, 2n, l}^{\Lambda, \Lambda_0} (\otimes_{i=1}^3 \mathcal{O}_{A_i}; p_1, \dots, p_{2n}) = \\
& = \binom{2n+2}{2} \int_k \dot{C}^\Lambda(k) \partial_{\vec{p}}^w \mathcal{L}_{\mathbf{D}, 2n+2, l-1}^{\Lambda, \Lambda_0} (\otimes_{i=1}^3 \mathcal{O}_{A_i}; k, -k, p_1, \dots, p_{2n}) \\
& - 4 \sum_{\substack{l_1+l_2=l \\ n_1+n_2=n+1}} n_1 n_2 \mathbb{S} \left[ \sum_{w_1+w_2+w_3=w} c_{\{w_j\}} \partial_{\vec{p}}^{w_1} \mathcal{L}_{\mathbf{D}, 2n_1, l_1}^{\Lambda, \Lambda_0} (\otimes_{i=1}^3 \mathcal{O}_{A_i}; q, p_1, \dots, p_{2n_1-1}) \right. \\
& \quad \times \partial_{\vec{p}}^{w_2} \dot{C}^\Lambda(q) \partial_{\vec{p}}^{w_3} \mathcal{L}_{2n_2, l_2}^{\Lambda, \Lambda_0} (p_{2n_1}, \dots, p_{2n}) \\
& + \sum_{w_1+w_2=w} c_{\{w_j\}} \int_k \partial_{\vec{p}}^{w_1} \mathcal{L}_{D_{\{1,2\}}, 2n_1, l_1}^{\Lambda, \Lambda_0} (\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}; k, p_1, \dots, p_{2n_1-1}) \\
& \quad \times \dot{C}^\Lambda(k) \partial_{\vec{p}}^{w_2} \mathcal{L}_{2n_2, l_2}^{\Lambda, \Lambda_0} (\mathcal{O}_{A_3}; -k, p_{2n_1}, \dots, p_{2n}) \left. \right] \tag{4.120}
\end{aligned}$$

The proof utilizes the same general strategy as the proof of theorem 3. In fact, to prove that the bound (4.119) is satisfied by the first two terms on the r.h.s. of the FE (4.120), we can mimic the proof of theorem 3, making the following adjustments:

- Replace  $D$  by  $D_{\{1,2,3\}}$ .
- Replace  $1/|x|^{D'-D}$  with  $1/|x_1 - x_2|^{\delta_1} \cdot 1/\max(|x_1|, |x_2|)^{\delta_2}$
- Replace  $d(2, n, l, w, D')$  by  $d(3, n, l, w, D')$ .

The first two changes do not affect the proof given in theorem 3. To see that the third one does not cause any problems either, we note that  $d(3, n, l, w, D')$  satisfies the following inequalities:

$$d(3, n+1, l-1, w, D') \leq d(3, n, l, w, D') \tag{4.121}$$

$$d(3, n_1, l_1, w, D') \leq d(3, n, l, w, D') \quad \text{if } n_1 + n_2 = n+1, l_1 + l_2 = l \text{ and } n_2 + l_2 \geq 1 \tag{4.122}$$

$$d(3, n, l, w + \tilde{w}, D') + |\tilde{w}| \leq d(3, n, l, w, D') \quad \text{for } |\tilde{w}| \leq D' + 1 - 2n - |w| \tag{4.123}$$

Those were the only properties of  $d$  that we needed in the part of the proof of theorem 3 regarding the first two terms on the r.h.s. of the FE (4.70). Thus, following the same computational steps as in the proof of theorem 3, we find that the contributions from the first two terms on the r.h.s. of the FE (4.120) satisfy the bound (4.119). We spare the reader the lengthy but straightforward repetitions of these calculations and instead give the details how to find a suitable estimate for the source terms, i.e. the last line of the FE (4.120).

**Lemma 2.** *Let  $x_3 = 0$  and fix regularization parameters such that  $\delta_1 := [A_1] + [A_2] - D_{\{1,2\}} \geq 0$  and  $\delta_2 := D_{\{1,2\}} + [A_3] - D_{\{1,2,3\}} \geq 0$ . Then we have the bound*

$$\begin{aligned}
& \left| \partial_{\vec{p}}^w \int_k \mathcal{L}_{D_{\{1,2\}}, 2n_1, l_1}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}; k, p_1, \dots, p_{2n_1-1}) \dot{C}^\Lambda(k) \mathcal{L}_{2n_2, l_2}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_3}; -k, p_{2n_1}, \dots, p_{2n}) \right| \\
& \leq M K_0^{(4n+8l-3)(|w|+D'-D_{\{1,2,3\}})} K_0^{D'(n+2l)^3} \prod_{i=1}^3 \sqrt{|v_i|!} \Lambda^{D_{\{1,2,3\}}-2n-|w|-1} e^{-m^2/\Lambda^2} \\
& \quad \times \frac{\sqrt{(|w|+D'-D_{\{1,2,3\}})!}}{|x_1-x_2|^{\delta_1} \max(|x_1|, |x_2|)^{\delta_2}} \sum_{\mu=0}^{d(N=3, n, l, w, D')} \frac{1}{\sqrt{\mu!}} \left( \frac{|\vec{p}|}{\Lambda} \right)^\mu \sum_{\lambda=0}^{2l+n} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}) + \sqrt{\lambda!})}{2^\lambda \lambda!}
\end{aligned} \tag{4.124}$$

where  $M = 5^{|w|+\delta_2} 2^{2l+n+1} 2^{2d}(2l+n+1) d 2^{\lfloor |w|+\delta_2 \rfloor/2}$  and where  $K_0$  is the constant (called  $K$  there) from our bound on the CAG's with two insertions, see thm 3.

*Proof of lemma 2.* The proof is similar to that of lemma 1 above. Assume for the moment that  $|x_2| > |x_1|$ . To begin with, we make use of the translation covariance property [see eq.(3.33)] of the CAG's and perform partial integrations in order to put the left hand side of (4.124) into the form

$$\begin{aligned}
& \left| \partial_{\vec{p}}^w \int_k \mathcal{L}_{D_{\{1,2\}}, 2n_1, l_1}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}(x_1) \otimes \mathcal{O}_{A_2}(x_2); k, p_1, \dots, p_{2n_1-1}) \right. \\
& \quad \left. \times \dot{C}^\Lambda(k) \mathcal{L}_{2n_2, l_2}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_3}(0); -k, p_{2n_1}, \dots, p_{2n}) \right| \\
& \leq \left| \int_k \sum_{w_1+w_2+w_3=w} c_{\{w_i\}} e^{ikx_2} e^{ix_2(p_1+\dots+p_{2n_1-1})} \right. \\
& \quad \left. \partial_k^{w_3} \left( \partial_{\vec{p}}^{w_1} \mathcal{L}_{D_{\{1,2\}}, 2n_1, l_1}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}(x_1-x_2) \otimes \mathcal{O}_{A_2}(0); k, p_1, \dots, p_{2n_1-1}) \right. \right. \\
& \quad \left. \left. \times \dot{C}^\Lambda(k) \partial_{\vec{p}}^{w_2} \mathcal{L}_{2n_2, l_2}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_3}(0); -k, p_{2n_1}, \dots, p_{2n}) \right) \right|
\end{aligned} \tag{4.125}$$

In analogy to the discussion following eq.(4.85) we can multiply by  $(2\|x_2\|)^{\delta_2}$ , where we recall the notation  $\|x\| = \max_{\alpha \in \{1, \dots, 4\}} |x_\alpha|$ , and write this factor as a  $k$ -derivative of  $e^{ikx_2}$ . This  $\delta_2$ -fold derivative can then be moved onto the terms in the second line of (4.125) by partial integration. We can then further substitute the bounds for the CAG's with one [cf. (4.66)]

and two insertions (cf. theorem 3) into the resulting expression. After some bookkeeping, this yields the estimate

$$\begin{aligned}
(4.125) \times (2\|x_2\|)^{\delta_2} &\leq M_0 K_0^{(4n+8l-3)(|w|+\delta_1+\delta_2)} K_0^{D'(n+2l)^3} \sqrt{(|w|+\delta_1+\delta_2)!|v_1|!|v_2|!} \\
&\times \frac{\Lambda^{D'-D-2n-|w|-5}}{|x_1-x_2|^{\delta_1}} e^{-m^2/\Lambda^2} \int_k e^{-k^2/2\Lambda^2} \sum_{\mu=0}^{d_{12}+d_3} \frac{1}{\sqrt{\mu!}} \left( \frac{|\vec{p}|+|k|}{\Lambda} \right)^\mu \sum_{\lambda=0}^{2l+n} \frac{\log^\lambda(\sup(\frac{|\vec{p}|+|k|}{\kappa}, \frac{\kappa}{m})}{2^\lambda \lambda!}
\end{aligned} \tag{4.126}$$

where  $M_0 = 5^{|w|+\delta_2} 2^{2l+n+1} 2^{d_{12}+d_3} (2l+n+1) d 2^{\lfloor |w|+\delta_2 \rfloor/2}$  and where we used the shorthand  $d_{12} = d(2, n_1, l_1, w_1, [A_1] + [A_2])$  and  $d_3 = d(1, n_2, l_2, w_2, [A_3])$ . We can replace the upper limit of summation over  $\mu$  by  $d(3, n, l, w, D')$ , since

$$d(3, n, l, w, D') = 2D'(n+l+4) + \sup(D'+1-2n-|w|, 0) \geq 2D'(n+l+3) + 2D' \geq d_{12} + d_3. \tag{4.127}$$

The momentum integral can then be estimated as in (4.89), and after division by  $(2\|x_2\|)^{\delta_2}$  and recalling that  $|x_2| \leq 2\|x_2\|$  we obtain the bound claimed in lemma 2 for  $|x_2| > |x_1|$ . If instead  $|x_2| < |x_1|$  we can just repeat the discussion above with the indices 1 and 2 exchanged, which completes the proof.  $\square$

With this lemma at hand we can verify that the bound claimed in theorem 4 is satisfied by the  $\Lambda'$  integral over the source terms. Indeed this integral can be bounded in the same manner as in the discussion of the source terms for the CAG's with two insertions:

**(A) Irrelevant terms** ( $2n + |w| > D_{\{1,2,3\}}$ ): Following the same computational steps as in the proof of theorem 3 [cf. (4.91)] we find that the contribution from the source term is bounded by (4.119), provided  $K$  is chosen large enough that

$$40(l+1)(n+1)^3 M K_0^{(4n+8l-3)(|w|+D'-D_{\{1,2,3\}})} K_0^{D'(n+2l)^3} \leq \frac{1}{8} K^{(4n+8l-3)(|w|+D'-D_{\{1,2,3\}})} K^{D'(n+2l)^3}. \tag{4.128}$$

**(B) Relevant terms** ( $2n + |w| \leq D_{\{1,2,3\}}$ ) at  $\vec{p} = \vec{0}$ : Proceeding as in (4.101) and the subsequent discussion, we find that the contribution from the source term is bounded by (4.119), provided  $K$  is chosen large enough that

$$\begin{aligned}
24(l+1)(n+1)^3 (2l+n+1) M K_0^{(4n+8l-3)(|w|+D'-D_{\{1,2,3\}})} K_0^{D'(n+2l)^3} \\
\leq \frac{1}{8} K^{(4n+8l-3)(|w|+D'-D_{\{1,2,3\}})} K^{D'(n+2l)^3}.
\end{aligned} \tag{4.129}$$

**(C) Relevant terms** ( $2n + |w| \leq D_{\{1,2,3\}}$ ) at  $\vec{p} \neq \vec{0}$ : Using the Taylor expansion with remainder technique as in the proof of theorem 3 [cf. eq.(4.104) and subsequent discussion],

we find that the contribution from the source term is bounded by (4.119), provided  $K$  is chosen large enough that

$$\begin{aligned} \sum_{j \leq D+1-2n-|w|} 48M(2l+n)(l+1)(n+1)^3 K_0^{(4n+8l-3)(|w|+j+D'-D)+D'(2l+n)^3} \\ \times (8n)^j j 2^{\frac{d(2,n,l,w,D')}{2}} 2^{\frac{j+|w|+D'-D}{2}} \leq K^{(4n+8l-3)(|w|+D'-D)+D'(2l+n)^3} \end{aligned} \quad (4.130)$$

Since all these inequalities hold for a sufficiently large choice of  $K$ , we have finished the proof of the theorem.  $\square$

Since we eventually want to remove the cutoffs, i.e. take the limits  $\Lambda \rightarrow 0, \Lambda_0 \rightarrow \infty$ , the following result will be useful:

**Corollary 2.** *Let  $x_3 = 0$  and choose  $\mathbf{D}, \delta_1$  and  $\delta_2$  as in proposition 1 and let  $\Lambda \leq m$ . There exists a constant  $K > 0$  such that*

$$\begin{aligned} |\partial_{\vec{p}}^w \mathcal{L}_{\mathbf{D}, 2n, l}^{\Lambda, \Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}(x_i); \vec{p})| &\leq K^{(4n+8l-3)(|w|+D'-D_{\{1,2,3\}})} K^{D'(n+2l)^3} \\ &\times \frac{m^{D_{\{1,2,3\}}-2n-|w|}}{|x_1-x_2|^{\delta_1} \max(|x_1-x_3|, |x_2-x_3|)^{\delta_2}} \sqrt{(2n+|w|-D_{\{1,2,3\}})_+!} \\ &\times \sqrt{|v_1|!|v_2|!|v_3|!(|w|+D'-D_{\{1,2,3\}})!} \sum_{\mu=0}^{d(3,n,l,w,D')} \left(\frac{|\vec{p}|}{m}\right)^\mu \sum_{\lambda=0}^{2l+n+1} \frac{\log_+^\lambda(\frac{|\vec{p}|}{m})}{2^\lambda \lambda!} \end{aligned} \quad (4.131)$$

with  $d(N, n, l, w, D') := 2D'(n+l+2(N-1)) + \sup(D'+1-2n-|w|, 0)$ , and  $A_i \equiv \{n_i, v_i\}$ .

**Remark:** This bound implies convergence of the multivariate Taylor expansion of the CAG's with three insertions in a neighborhood of  $\vec{x} = (x_1, x_2, x_3)$  (where  $x_i \neq x_j$ ) for any choice of regularization  $\mathbf{D}$ . This can be seen from the fact that the bound grows like

$$j! \left( \sup(|\vec{p}|/m, 1)^{2(n+l+4)+1} \tilde{K} / \min_{i \neq j} |x_i - x_j| \right)^j \quad (4.132)$$

if we take  $j$  derivatives with respect to  $\vec{x} = (x_1, x_2, x_3)$ . [Here we also used the Lowenstein rule (3.47).] Consequently, the CAG's with or without cutoffs are analytic functions of their spacetime arguments for non-coinciding configurations. Since the OPE coefficients are derived from the CAG's [see defn. 1], the same holds true for them.

Corollary 2 follows by inserting the bound from theorem 1 once more into the FE and integrating over  $\Lambda$  between 0 and  $m$  (cf. proof of corollary 1).

#### 4.4 Amputated Greens functions with $N = 3$ insertions

We now give bounds for the regulated AG's with three insertions,  $G_D^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2} \otimes \mathcal{O}_{A_3})$ , see eq. (3.59), or more precisely, certain combinations of Taylor coefficients of their moments  $\mathcal{G}_{n,l,D}^{\Lambda, \Lambda_0}$ , where we recall the definition of the Taylor expansion operator in eq. (3.61). The result is:

**Theorem 5.** Let  $D = [A_1] + [A_2] + [A_3] + \Delta$ , where  $\Delta \geq 0$ , and let  $\Lambda \leq m$ . Then

$$\begin{aligned}
& \left| \left( 1 - \sum_{j \leq \Delta} \mathbb{T}_{(x_1, x_2, x_3) \rightarrow (x_3, x_3, x_3)}^j \right) \mathcal{G}_{D, 2n, l}^{\Lambda, \Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}(x_i); \vec{p}) \right| \\
& \leq m^{-2n-1} \frac{(\tilde{K} m \max(|x_1 - x_2|, |x_2 - x_3|, |x_1 - x_3|))^{\sum [A_i] + 1 + \Delta}}{\min(|x_1 - x_2|, |x_2 - x_3|, |x_1 - x_3|)^{\sum [A_i] + 1}} \\
& \quad \times \sup(1, \frac{|\vec{p}|}{m})^{d(3, n, l, w=0, D+D')} \frac{\prod_{i=1}^3 [A_i]!}{\sqrt{\Delta!}} \sum_{\lambda=0}^{2l+n+1} \frac{\log_{+}^{\lambda}(\frac{|\vec{p}|}{m})}{2^{\lambda} \lambda!}
\end{aligned} \tag{4.133}$$

where  $\tilde{K} > 0$  is a constant depending on  $n$  and  $l$ , and where  $A_i \equiv \{n_i, v_i\}$ .

*Proof.* Note that, by eq. (3.59), the quantities  $G_D^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2} \otimes \mathcal{O}_{A_3})$  and  $F_D^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2} \otimes \mathcal{O}_{A_3})$  only differ by products of CAG's with one insertion, for which we already have given corresponding bounds in sec. 4.1. Thus, the task boils down to bounds on  $(1 - \sum_{j \leq \Delta} \mathbb{T}_{(x_1, x_2, x_3) \rightarrow (x_3, x_3, x_3)}^j) \mathcal{F}_{n, l, D}^{\Lambda, \Lambda_0}$ . Although this quantity satisfies a FE, we have not been able to derive our bounds straightforwardly from that FE in the manner described above. The technical obstruction seems to be that one cannot make use of the partial integration trick that has been applied in all previous proofs to bound the momentum integrals, see e.g. eq.(4.85). This is due to the fact that now we have more than just two CAG's in the source terms, but only two of them are integrated over.

To get around this, we will prove a decomposition of  $F_D^{\Lambda, \Lambda_0}$  (and correspondingly its moments  $\mathcal{F}_{D, n, l}^{\Lambda, \Lambda_0}$ ), into quantities that we already know how to estimate.

**Lemma 3** (Decomposition Lemma). Let  $x_3 = 0$ . For any  $D \geq -1$  the functionals  $F_D^{\Lambda, \Lambda_0}$  can be decomposed as

$$\left( 1 - \sum_{j \leq D-D'} \mathbb{T}_{(x_1, x_2) \rightarrow (0, 0)}^j \right) F_D^{\Lambda, \Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}) = F_1 + F_2 + F_3, \tag{4.134}$$

where  $F_1, F_2, F_3$  are the functions of  $x_1, x_2$  given by

$$\begin{aligned}
F_1 = & \left( 1 - \sum_{j \leq D-[A_1]} \mathbb{T}_{(x_1, x_2) \rightarrow (0, x_2)}^j \right) \\
& \times \left[ L_{(D_{\{1, 2, 3\}}=D, D_{\{2, 3\}}=-1)}^{\Lambda, \Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}) - L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_2} \otimes \mathcal{O}_{A_3}) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}) \right] \\
& + \sum_{j_1 \leq D-[A_1]} \mathbb{T}_{(x_1, x_2) \rightarrow (0, x_2)}^{j_1} \left( 1 - \sum_{j_2 \leq D-D'-j_1} \mathbb{T}_{(x_1, x_2) \rightarrow (x_1, 0)}^{j_2} \right) \\
& \times \left[ L_{(D_{\{1, 2, 3\}}=D, D_{\{2, 3\}}=D-[A_1]-j_1)}^{\Lambda, \Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}) - L_{D-[A_1]-j_1}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_2} \otimes \mathcal{O}_{A_3}) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}) \right].
\end{aligned} \tag{4.135}$$

The function  $F_2$  is given by the same expression replacing  $A_1 \rightarrow A_2, D_{\{2,3\}} \rightarrow D_{\{1,3\}}$  and  $\mathbb{T}_{(x_1, x_2) \rightarrow (0, x_2)} \rightarrow \mathbb{T}_{(x_1, x_2) \rightarrow (x_1, 0)}$ . The function  $F_3$  is given by

$$\begin{aligned}
F_3 = & \left(1 - \sum_{j \leq D-[A_1]} \mathbb{T}_{\left(\frac{x_1+x_2}{\sqrt{2}}, \frac{x_1-x_2}{\sqrt{2}}\right) \rightarrow (0, \frac{x_1-x_2}{\sqrt{2}})}^j\right) \\
& \times \left[ L_{(D_{\{1,2,3\}}=D, D_{\{1,2\}}=-1)}^{\Lambda, \Lambda_0} (\otimes_{i=1}^3 \mathcal{O}_{A_i}) - L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_3}) \right] \\
& + \sum_{j_1 \leq D-[A_3]} \mathbb{T}_{\left(\frac{x_1+x_2}{\sqrt{2}}, \frac{x_1-x_2}{\sqrt{2}}\right) \rightarrow (0, \frac{x_1-x_2}{\sqrt{2}})}^{j_1} \left(1 - \sum_{j_2 \leq D-D'-j_1} \mathbb{T}_{\left(\frac{x_1+x_2}{\sqrt{2}}, \frac{x_1-x_2}{\sqrt{2}}\right) \rightarrow (\frac{x_1+x_2}{\sqrt{2}}, 0)}^{j_2}\right) \\
& \times \left[ L_{(D_{\{1,2,3\}}=D, D_{\{1,2\}}=D-[A_3]-j_1)}^{\Lambda, \Lambda_0} (\otimes_{i=1}^3 \mathcal{O}_{A_i}) - L_{D-[A_3]-j_1}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_3}) \right].
\end{aligned} \tag{4.136}$$

Here it is understood that we express the CAG's as functions of  $(x_1 + x_2, x_1 - x_2)$ .

**Remark:** The CAG's are of course translation invariant; the restriction to  $x_3 = 0$  is made only for convenience here.

The lemma can be proved by an induction in  $D$ , using at each step that both sides satisfy the same FE and boundary conditions, and using (4.117). We omit the proof here. Continuing the proof of theorem 5, we now expand both sides of the lemma into a power series in  $\varphi$  and  $\hbar$ . Recall also that the remainder of a Taylor expansion can be expressed as the following integral:

$$\begin{aligned}
(1 - \sum_{j \leq \Delta} \mathbb{T}_{\vec{x} \rightarrow \vec{y}}^j) f(x_1, \dots, x_N) = \\
\sum_{|v|=\Delta+1} \frac{(\vec{x} - \vec{y})^v}{\Delta!} \int_0^1 d\tau (1 - \tau)^\Delta \partial^v \left[ f(y_1 + \tau(x_1 - y_1), \dots, y_N - \tau(x_N - y_N)) \right].
\end{aligned} \tag{4.137}$$

Using that formula in the decomposition lemma and applying the Lowenstein rules, eqs. (3.46), (3.47) and (3.48), as well as the previous bounds on the CAG's with insertions, i.e. (4.68) and corollaries 1 and 2, we obtain straightforwardly the bound stated in the theorem, but on  $\mathcal{F}_{n,l,D}^{\Lambda, \Lambda_0}$ , rather than  $\mathcal{G}_{n,l,D}^{\Lambda, \Lambda_0}$ . As we said, these only differ by products of CAG's with one insertion, for which we have the bound (4.66) again. This yields the statement of the theorem, after using also translation invariance in order to proceed to  $x_3 \neq 0$ . As an

illustrative example, we estimate a term in  $F_3$ . Let  $\mathbf{D} = (D_{\{1,2,3\}} = D, D_{\{1,2\}} = -1)$ .

$$\begin{aligned}
& \left| \left( 1 - \sum_{j \leq D - [A_3]} \mathbb{T}_{\left(\frac{x_1+x_2}{\sqrt{2}}, \frac{x_1-x_2}{\sqrt{2}}\right) \rightarrow (0, \frac{x_1-x_2}{\sqrt{2}})}^j \right) \mathcal{L}_{\mathbf{D}, 2n, l}^{\Lambda, \Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}; \vec{p}) \right| \\
& \leq \frac{|x_1 + x_2|^{D - [A_3] + 1}}{(D - [A_3])!} \sum_{|w| = D + 1 - [A_3]} \left| \int_0^1 d\tau (1 - \tau)^{|w| - 1} \partial_{\tau}^w \left( \frac{x_1 + x_2}{2} \right) \right. \\
& \quad \times \left. \left( e^{i\tau \frac{x_1 + x_2}{2} (p_1 + \dots + p_{2n})} \mathcal{L}_{\mathbf{D}, 2n, l}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}(\frac{x_1 - x_2}{2}) \otimes \mathcal{O}_{A_2}(-\frac{x_1 - x_2}{2}) \otimes \mathcal{O}_{A_3}(-\tau \frac{x_1 + x_2}{2}); \vec{p}) \right) \right| \\
& \leq \frac{|x_1 + x_2|^{D - [A_3] + 1}}{(D - [A_3])!} \sum_{|w_1 + w_2| = D + 1 - [A_3]} c_{\{w_j\}} |\vec{p}|^{|w_1|} \left| \mathcal{L}_{\mathbf{D}, 2n, l}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}(\frac{x_1 - x_2}{2}) \otimes \mathcal{O}_{A_2}(-\frac{x_1 - x_2}{2}) \otimes \partial^{w_2} \mathcal{O}_{A_3}(0); \vec{p}) \right|
\end{aligned} \tag{4.138}$$

Here we expressed the remainder of the Taylor expansion through eq.(4.137) and made use of the covariance property of the CAG's, eq.(3.33). Using the bound on the CAG's with three insertions given in corollary 2, we obtain:

$$\begin{aligned}
& \left| \left( 1 - \sum_{j \leq D - [A_3]} \mathbb{T}_{\left(\frac{x_1+x_2}{\sqrt{2}}, \frac{x_1-x_2}{\sqrt{2}}\right) \rightarrow (0, \frac{x_1-x_2}{\sqrt{2}})}^j \right) \mathcal{L}_{\mathbf{D}, 2n, l}^{\Lambda, \Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}; \vec{p}) \right| \\
& \leq \frac{|x_1 + x_2|^{D - [A_3] + 1}}{(D - [A_3])!} 2^{D + 1 - [A_3]} K^{(D' + D - [A_3] + 1)(2l + n)^3 + (4n + 8l - 3)([A_1] + [A_2] + 1)} \frac{m^{D - 2n} \sqrt{(2n - D)_+!}}{|x_1 - x_2|^{[A_1] + [A_2] + 1}} \\
& \times \sqrt{|v_1|! |v_2|! (|v_3| + D - [A_3] + 1)! ([A_1] + [A_2] + 1)!} \left( \sup(1, \frac{\vec{p}}{m}) \right)^{d(3, n, l, 0, D' + D - [A_3] + 1)} \sum_{\lambda=0}^{2l + n + 1} \frac{\log_{+}^{\lambda}(\frac{|\vec{p}|}{m})}{2^{\lambda} \lambda!} \\
& \leq m^{-2n - 1} \frac{\left( \tilde{K} m \max(|x_1|, |x_2|) \right)^{D + 1}}{\min(|x_1 - x_2|, |x_1|, |x_2|)^{D' + 1}} \frac{\prod_{i=1}^3 [A_i]!}{\sqrt{(D - [A_3])!}} \\
& \times \left( \sup(1, \frac{\vec{p}}{m}) \right)^{d(3, n, l, 0, D' + D - [A_3] + 1)} \sum_{\lambda=0}^{2l + n + 1} \frac{\log_{+}^{\lambda}(\frac{|\vec{p}|}{m})}{2^{\lambda} \lambda!}
\end{aligned} \tag{4.139}$$

where  $\tilde{K}$  is some constant depending on  $n$  and  $l$ . Using translation covariance we obtain a bound compatible with eq.(4.133) also for  $x_3 \neq 0$ . The other terms on the r.h.s. of the decomposition lemma can be bounded in a similar fashion.  $\square$

## 5 Proof of thm. 1

We repeat the statement of the theorem for convenience:



**Theorem 1.** Let  $f_p(x)$  be any smooth function on  $x \in \mathbb{R}^4$  such that the support of the Fourier transform  $\hat{f}_p(q)$  is contained in a ball  $|p - q| \leq 1$ . Define the smeared spectator fields by

$$\varphi(f_p) \equiv \int d^4x \varphi(x) f_p(x) .$$

Then the remainder of the OPE, carried out up to operators of dimension  $D = \sum_{i=1}^3 [A_i] + \Delta$ , at  $l$ -loops, is bounded by

$$\begin{aligned} & \left| \left\langle \mathcal{O}_{A_1}(x_1) \mathcal{O}_{A_2}(x_2) \mathcal{O}_{A_3}(x_3) \varphi(f_{p_1}) \cdots \varphi(f_{p_n}) \right\rangle \right. \\ & \quad \left. - \sum_{C: [C] \leq D} \mathcal{C}_{A_1 A_2 A_3}^C(x_1, x_2, x_3) \left\langle \mathcal{O}_C(x_3) \varphi(f_{p_1}) \cdots \varphi(f_{p_n}) \right\rangle \right| \\ & \leq m^{n-1} \prod_{i=1}^3 [A_i]! \prod_j \sup |\hat{f}_{p_j}| \sup(1, \frac{|\vec{p}|_n}{m})^{(4 \sum [A_i] + 2\Delta)(n+l+9/2)+3n} \\ & \quad \times \sum_{\lambda=0}^{2l+n/2+1} \frac{\log^\lambda \sup(1, \frac{|\vec{p}|_n}{m})}{2^\lambda \lambda!} \\ & \quad \times \frac{1}{\sqrt{\Delta!}} \frac{(\tilde{K} m \max(|x_1 - x_2|, |x_2 - x_3|, |x_1 - x_3|))^{\sum [A_i] + 1 + \Delta}}{(\min(|x_1 - x_2|, |x_2 - x_3|, |x_1 - x_3|))^{\sum [A_i] + 1}} \end{aligned} \quad (5.140)$$

in  $g\varphi^4$ -theory. Here  $[A]$  denotes the canonical dimension of a composite field  $\mathcal{O}_A$  as in eq. (1.3).  $\tilde{K}$  is a constant depending on  $n$  and  $l$ .

*Proof.* A part of the proof can be generalized to  $N$  field insertions without any additional effort, so we will keep  $N \geq 2$  arbitrary for as long as possible. Define the remainder functional

$$R_D^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) := G^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) - \sum_{C: [C] \leq D} \mathcal{C}_{A_1 \dots A_N}^C(x_1, \dots, x_{N-1}) L^{\Lambda, \Lambda_0}(\mathcal{O}_C(x_N)) \quad (5.141)$$

which allows us to write (for the theory with UV and IR cutoffs  $\Lambda_0$  and  $\Lambda$ )

$$\begin{aligned} & \left| \left\langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_{N-1}}(x_{N-1}) \mathcal{O}_{A_N}(x_N) \varphi(f_{p_1}) \cdots \varphi(f_{p_n}) \right\rangle \right. \\ & \quad \left. - \sum_{C: [C] - D' \leq \Delta} \mathcal{C}_{A_1 \dots A_N}^C(x_1, \dots, x_N) \left\langle \mathcal{O}_C(x_N) \varphi(f_{p_1}) \cdots \varphi(f_{p_n}) \right\rangle \right| \\ & = \sum_{\substack{I_1 \cup \dots \cup I_j = \{1, \dots, n\} \\ l_1 + \dots + l_j = l}} \int_{\vec{q}} \mathcal{R}_{D, |I_1|, l_1}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i); \vec{q}_{I_1}) \bar{\mathcal{L}}_{|I_2|, l_2}^{\Lambda, \Lambda_0}(\vec{q}_{I_2}) \cdots \bar{\mathcal{L}}_{|I_j|, l_j}^{\Lambda, \Lambda_0}(\vec{q}_{I_j}) \prod_{i=1}^n C^{\Lambda, \Lambda_0}(q_i) \hat{f}_{p_i}(q_i) \end{aligned} \quad (5.142)$$

where  $\bar{\mathcal{L}}_{n,l}^{\Lambda, \Lambda_0}$  are the expansion coefficients of the generating functional  $\bar{L}^{\Lambda, \Lambda_0}(\varphi) = L^{\Lambda, \Lambda_0}(\varphi) + \frac{1}{2} \langle \varphi, (C^{\Lambda, \Lambda_0})^{-1} \star \varphi \rangle$  without the momentum conservation delta functions taken out. We wish

to find a bound for the above expression. Since we already have bounds on  $\mathcal{L}_{n,l}^{\Lambda,\Lambda_0}$  from (4.63), and since  $C^{\Lambda,\Lambda_0}$  can be estimated trivially as  $C^{\Lambda,\Lambda_0}(p) \leq [\sup(m, |p|)]^{-2}$ , we will be concerned with  $\mathcal{R}_{D,n,l}^{\Lambda,\Lambda_0}$  in the following. The following lemma will allow us to express  $\mathcal{R}_{D,n,l}^{\Lambda,\Lambda_0}$  in terms of quantities with known bounds as given in the previous sections:

**Lemma 4.** *The remainder functionals satisfy*

$$R_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) = (1 - \sum_{j \leq \Delta} \mathbb{T}_{\vec{x} \rightarrow (x_N, \dots, x_N)}^j) G_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) \quad (5.143)$$

with  $\Delta = D - \sum_{i=1}^N [A_i]$ .

This lemma reduces to lemma 4.1 of [4] in the case  $N = 2$ . The proof is given in Appendix A.

Hence, in order to bound the remainder of the OPE we have to find an estimate on the remainder of the Taylor expansion of the regularized AG's. Here we have theorem 5 in the  $N = 3$  case. Substituting this bound along with the bound (4.67) on the CAG's without operator insertions into eq.(5.142), and also using  $C^{\Lambda,\Lambda_0}(p) \leq [\sup(m, |p|)]^{-2}$ , we obtain the statement of the theorem (note that the resulting bound is independent of  $\Lambda \leq m$  and  $\Lambda_0$ , so the cutoffs can be removed safely).  $\square$

## 6 Proof of thm. 2

For convenience, we repeat the statement of this theorem as well:

**Theorem 2.** *Up to any arbitrary but fixed loop order  $l$  in  $g\varphi^4$ -theory, the identity*

$$\mathcal{C}_{A_1 A_2 A_3}^B(x_1, x_2, x_3) = \sum_C \mathcal{C}_{A_1 A_2}^C(x_1, x_2) \mathcal{C}_{A_3}^B(x_2, x_3) . \quad (6.144)$$

*holds for all configurations satisfying*

$$\frac{|x_1 - x_2|}{|x_2 - x_3|} < \frac{1}{\tilde{K}}$$

*for some (sufficiently large) constant  $\tilde{K} > 0$  (depending on  $l, B$ ).*

*Proof:* The proof has two main steps, which will be discussed in more detail below:

1. Show that the OPE still converges on the spacetime domain  $\frac{|x_1 - x_2|}{|x_2 - x_3|} < \frac{1}{\tilde{K}}$  when performed in successive steps.
2. Show that this implies the relation (6.144) on the level of OPE coefficients.

In the next two sections we will focus on these two issues.

## 6.1 Partial OPE

We have already shown that the remainder

$$G^{\Lambda, \Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}) - \sum_{[C] \leq D} \mathcal{C}_{A_1 A_2 A_3}^C L^{\Lambda, \Lambda_0}(\mathcal{O}_C) \quad (6.145)$$

goes to zero as  $D \rightarrow \infty$ . Instead of expanding all three operator insertions, we now consider a similar expansion in just two of these operators, say  $\mathcal{O}_{A_1}$  and  $\mathcal{O}_{A_2}$ , and leave the other one,  $\mathcal{O}_{A_3}$ , untouched, namely the expression

$$G^{\Lambda, \Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}(x_i)) - \sum_{[C] \leq D} \mathcal{C}_{A_1 A_2}^C(x_1 - x_2) G^{\Lambda, \Lambda_0}(\mathcal{O}_C(x_2) \otimes \mathcal{O}_{A_3}(x_3)) . \quad (6.146)$$

The following lemma will allow us to bound the remainder of this partial OPE.

**Lemma 5.** *Fix  $D \geq -1$ , and define regularization parameters as*

$$\mathbf{D} = (D_{\{1,2,3\}} = -1, D_{\{2,3\}} = -1, D_{\{1,3\}} = -1, D_{\{1,2\}} = D) .$$

*The remainder of the partial OPE can be expressed as*

$$\begin{aligned} & G^{\Lambda, \Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}(x_i)) - \sum_{[C] \leq D} \mathcal{C}_{A_1 A_2}^C(x_1 - x_2) G^{\Lambda, \Lambda_0}(\mathcal{O}_C(x_2) \otimes \mathcal{O}_{A_3}(x_3)) \\ &= (1 - \sum_{j=0}^{D-[A_1]-[A_2]} \mathbb{T}_{(x_1, x_2, x_3) \rightarrow (x_2, x_2, x_3)}^j) \\ & \quad \times \left[ L_{\mathbf{D}}^{\Lambda, \Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}) - L_D^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_3}) - L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_2} \otimes \mathcal{O}_{A_3}) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}) \right. \\ & \quad \left. - L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_3}) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_2}) + L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_2}) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_3}) \right] . \end{aligned} \quad (6.147)$$

The proof of this lemma can be found in appendix B. Lemma 5, combined with our bounds on the CAG's [see (4.66), theorem 3 and theorem 4], allows us to estimate the remainder of the partial OPE.

**Theorem 6.** *Assume  $|x_1 - x_2| \leq |x_2 - x_3|$  and let  $\Lambda \leq m$ . There exists a constant  $\tilde{K} > 0$*

depending on  $n$  and  $l$ , such that for all  $D - [A_1] - [A_2] = \Delta$

$$\begin{aligned}
& \left| \partial_{\vec{p}}^w \left( \mathcal{G}_{2n,l}^{\Lambda,\Lambda_0} (\otimes_{i=1}^3 \mathcal{O}_{A_i}(x_i); \vec{p}) - \sum_{[C] \leq D} \mathcal{C}_{A_1 A_2}^C(x_1 - x_2) \mathcal{G}_{2n,l}^{\Lambda,\Lambda_0} (\mathcal{O}_C(x_2) \otimes \mathcal{O}_{A_3}(x_3); \vec{p}) \right) \right| \\
& \leq m^{-2n-|w|-1} \tilde{K}^{|w|} |w|! \prod_{i=1}^3 [A_i]! \sum_{\lambda=0}^{2l+n+1} \frac{\log_+^\lambda(\frac{|\vec{p}|}{m})}{2^\lambda \lambda!} \sup(1, m|\vec{x}|)^{|w|} \\
& \quad \times \left( \frac{\tilde{K} \sup(\frac{|\vec{p}|}{m}, 1)^{2n+2l+5}}{\min(|x_2 - x_3|, |x_1 - x_3|, 1/m)} \right)^{D'+1} \\
& \quad \times \frac{\left( \tilde{K} \sup(\frac{|\vec{p}|}{m}, 1)^{2n+2l+5} |x_1 - x_2| \right)^\Delta}{\min(|x_2 - x_3|^{\Delta-1} \cdot \min(|x_2 - x_3|, |x_1 - x_3|), \sqrt{\Delta!}/m^\Delta)}
\end{aligned} \tag{6.148}$$

where  $D' = [A_1] + [A_2] + [A_3]$ .

*Proof.* We use lemma 5 to express the l.h.s. as the remainder of a Taylor expansion in  $x_1 - x_2$ . We write this Taylor expansion as  $(1 - \sum_{j \leq \Delta} \mathbb{T}^j) = (1 - \sum_{j \leq \Delta-1} \mathbb{T}^j) - \mathbb{T}^\Delta$  and apply again the well known formula (4.137) for the term in brackets on the r.h.s.. For the expressions on the r.h.s. of lemma 5 we can insert our bounds from (4.68), corollary 1 and corollary 2. Note that these bounds are always given for configurations where one spacetime argument is zero. Hence, in order to be able to use our bounds we also have to make use of the translation properties of the CAG's. For example, we can write

$$\mathcal{L}_{2n,l}^{\Lambda,\Lambda_0} (\mathcal{O}_{A_1}(x_1) \otimes \mathcal{O}_{A_3}(x_3); \vec{p}) = e^{ix_3(p_1 + \dots + p_{2n})} \mathcal{L}_{2n,l}^{\Lambda,\Lambda_0} (\mathcal{O}_{A_1}(x_1 - x_3) \otimes \mathcal{O}_{A_3}(0); \vec{p}) \tag{6.149}$$

and apply corollary 1 for the term on the right. Note that the momentum derivatives can either act on the CAG's or on the exponential factors. Derivatives on the latter simply give rise to powers of the spacetime variables. This is accounted for by the factor  $\sup(1, m|\vec{x}|)^{|w|}$  in the bound (6.148). When estimating the  $\tau$ -integral in the formula for remainder of the Taylor expansion we also make use of our assumption  $|x_1 - x_2| \leq |x_2 - x_3|$  to replace expressions like

$$\frac{(1 - \tau) |x_1 - x_2|}{|x_2 - x_3 + \tau(x_1 - x_2)|} = \frac{|x_1 - x_2|}{|x_2 - x_3|} \frac{1 - \tau}{|\frac{x_2 - x_3}{|x_2 - x_3|} + \tau \frac{x_1 - x_2}{|x_2 - x_3|}|} \leq \frac{|x_1 - x_2|}{|x_2 - x_3|} \tag{6.150}$$

under the integral from  $\tau = 0$  to  $\tau = 1$ . As an example, consider the following contribution

to the l.h.s. of (6.148):

$$\begin{aligned}
& \left| \partial_{\vec{p}}^w \left( 1 - \sum_{j=0}^{D-[A_1]-[A_2]} \mathbb{T}_{(x_1, x_2, x_3) \rightarrow (x_2, x_2, x_3)}^j \right) \right. \\
& \quad \times \mathcal{L}_{2n_1, l_1}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_3}; p_1, \dots, p_{2n_1-1}) \mathcal{L}_{2n_2, l_2}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_2}; p_{2n_1}, \dots, p_{2n}) \left. \right| \\
& \leq 2 \frac{|x_1 - x_2|^\Delta}{(\Delta - 1)!} \sum_{|v|=\Delta} \sup_{0 \leq \tau \leq 1} \left| \partial_{\vec{p}}^w (1 - \tau)^{\Delta-1} e^{ix_3(p_1 + \dots + p_{2n_1-1})} e^{ix_2(p_{2n_1} + \dots + p_{2n})} \right. \\
& \quad \mathcal{L}_{2n_1, l_1}^{\Lambda, \Lambda_0}(\partial^v \mathcal{O}_{A_1}(x_2 - x_3 + \tau(x_1 - x_2)) \otimes \mathcal{O}_{A_3}(0); p_1, \dots, p_{2n_1-1}) \mathcal{L}_{2n_2, l_2}^{\Lambda, \Lambda_0}(\mathcal{O}_{A_2}(0); p_{2n_1}, \dots, p_{2n}) \left. \right| \\
& \leq \sup_{0 \leq \tau \leq 1} \left( \frac{|1 - \tau|^{\Delta-1} |x_1 - x_2|^\Delta}{|x_2 - x_3 + \tau(x_1 - x_2)|^{\Delta+[A_1]+[A_3]+1}} \right) \frac{\prod_{i=1} [A_i]!}{\sqrt{[A_2]!}} \\
& \quad \times m^{[A_2]-2n-|w|-1} \tilde{K}^{|w|+D'+\Delta} |w|! \sup\left(\frac{|\vec{p}|}{m}, 1\right)^{d(3,n,l,w,D'+\Delta)} \sum_{\lambda=0}^{2l+n+1} \frac{\log_+^\lambda(\frac{|\vec{p}|}{m})}{2^\lambda \lambda!} \sup(1, m|\vec{x}|)^{|w|} \left| \right. \\
& \leq m^{-2n-|w|-1} \tilde{K}^{|w|} |w|! \prod_{i=1} [A_i]! \sum_{\lambda=0}^{2l+n+1} \frac{\log_+^\lambda(\frac{|\vec{p}|}{m})}{2^\lambda \lambda!} \sup(1, m|\vec{x}|)^{|w|} \\
& \quad \times \left( \frac{\tilde{K} \sup(\frac{|\vec{p}|}{m}, 1)^{2n+2l+5}}{\min(|x_2 - x_3|, |x_1 - x_3|)} \right)^{[A_1]+[A_3]+1} \left( \frac{\tilde{K} \sup(\frac{|\vec{p}|}{m}, 1)^{2n+2l+5}}{[A_2]!^{1/2} [A_2]/m} \right)^{[A_2]} \\
& \quad \times \left( \frac{\tilde{K} \sup(\frac{|\vec{p}|}{m}, 1)^{2n+2l+5} |x_1 - x_2|}{|x_2 - x_3|} \right)^{\Delta-1} \left( \frac{\tilde{K} \sup(\frac{|\vec{p}|}{m}, 1)^{2n+2l+5} |x_1 - x_2|}{\min(|x_1 - x_3|, |x_2 - x_3|)} \right)
\end{aligned} \tag{6.151}$$

Here we have used the known bounds for the CAG's with one and two insertions as well as the inequality (6.150). In a similar manner one checks that all the other terms on the r.h.s. of lemma 5 satisfy the bound (6.148) as well.  $\square$

**Remark:** Obviously, the r.h.s. of (6.148) vanishes as we take the limit  $\Delta \rightarrow \infty$  provided that

$$\left( \frac{\tilde{K} \sup(\frac{|\vec{p}|}{m}, 1)^{n+2l+5} |x_1 - x_2|}{|x_2 - x_3|} \right) < 1 \tag{6.152}$$

which defines an open spacetime region  $(x_1, x_2, x_3) \in \mathcal{S}(n, l, \vec{p}, m)$  for any finite values of  $n, l, m$  and  $\vec{p}$ . Hence, the partial OPE converges in that region, which is the upshot of this section.

## 6.2 Proof of Factorization

We are now ready to give the proof of the factorization identity, theorem 2. We have found in eq.(5.141) for  $N = 3$  that

$$G^{0,\Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}(x_i)) = \sum_{[C] \leq D} \mathcal{C}_{A_1 A_2 A_3}^C(x_1, x_2, x_3) L^{0,\Lambda_0}(\mathcal{O}_C(x_3)) + R_D^{0,\Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}(x_i)) \quad . \quad (6.153)$$

We may also write

$$\begin{aligned} G^{0,\Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}(x_i)) &= \sum_{[C_1] \leq D_1} \mathcal{C}_{A_1 A_2}^{C_1}(x_1, x_2) G^{0,\Lambda_0}(\mathcal{O}_{C_1}(x_2) \otimes \mathcal{O}_{A_3}(x_3)) + (\text{partial remainder}) \\ &= \sum_{[C_1] \leq D_1} \sum_{[C_2] \leq D_2} \mathcal{C}_{A_1 A_2}^{C_1}(x_1, x_2) \mathcal{C}_{C_1 A_3}^{C_2}(x_2, x_3) L^{0,\Lambda_0}(\mathcal{O}_{C_2}(x_3)) \\ &\quad + \sum_{[C_1] \leq D_1} \mathcal{C}_{A_1 A_2}^{C_1}(x_1, x_2) R_{D_2}^{0,\Lambda_0}(\mathcal{O}_{C_1}(x_2) \otimes \mathcal{O}_{A_3}(x_3)) \\ &\quad + (\text{partial remainder}) , \end{aligned} \quad (6.154)$$

where the ‘partial remainder’, i.e. the remainder of the partial OPE, is the expression (6.146) with  $D$  replaced by  $D_1$ . The key point is that this partial remainder is bounded by theorem 6.

We will exploit these two different ways of writing  $G^{0,\Lambda_0}$ . To do this, we firstly note the relation

$$\mathcal{D}^B \{R_D^{0,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i})\} = 0 \quad (6.155)$$

for  $D \geq [B]$ . To see this, we use lemma 4 to express the remainder  $R_D^{0,\Lambda_0}$  in terms of the regularized AG’s,  $G_D^{0,\Lambda_0}$ . Then we rewrite  $G_D^{0,\Lambda_0}$  using (3.59). According to the boundary conditions for the  $F_D^{\Lambda,\Lambda_0}$  functionals, eq.(3.56), we find that  $\mathcal{D}^B F_D^{0,\Lambda_0}$  vanishes for  $D \geq [B]$ . It remains to show that the contribution in (3.59) to the remainder from the product of CAG’s with one insertion vanishes, too. It reads explicitly

$$\begin{aligned} &(1 - \sum_{j \leq \Delta} \mathbb{T}_{\vec{x} \rightarrow \vec{y}=(x_N, \dots, x_N)}^j) \prod_{i=1}^N L^{0,\Lambda_0}(\mathcal{O}_{A_i}(x_i)) \\ &= \sum_{|v|=\Delta+1} \frac{(\vec{x} - \vec{y})^v}{(|v| - 1)!} \int_0^1 d\tau (1 - \tau)^{|v|-1} \partial_{\vec{y} + \tau(\vec{x} - \vec{y})}^v \prod_{i=1}^N L^{0,\Lambda_0}(\mathcal{O}_{A_i}(x_N + \tau(x_i - x_N))) \end{aligned} \quad (6.156)$$

where again we expressed the remainder of the Taylor expansion through an integral formula. Using the Lowenstein rule (3.46) to pull the derivatives into the CAG’s, it follows that this expression vanishes as we apply  $\mathcal{D}^B$  due to the boundary condition (3.37). This yields eq.(6.155).

Secondly, the boundary condition (3.37) implies

$$\mathcal{D}^B \sum_{[C] \leq D} \mathcal{C}_{A_1 \dots A_N}^C L^{0, \Lambda_0}(\mathcal{O}_C) = \mathcal{D}^B \sum_{[C] < [B]} \mathcal{C}_{A_1 \dots A_N}^C L^{0, \Lambda_0}(\mathcal{O}_C) + \mathcal{C}_{A_1 \dots A_N}^B \quad (6.157)$$

for  $D \geq [B]$ . Thus, applying the operator  $\mathcal{D}^B$  to (6.153) and (6.154) and choosing  $D, D_2 \geq [B]$ , we obtain

$$\begin{aligned} & \mathcal{D}^B \sum_{[C] \leq [B]} \mathcal{C}_{A_1 A_2 A_3}^C(x_1, x_2, x_3) L^{0, \Lambda_0}(\mathcal{O}_C(x_3)) + \mathcal{C}_{A_1 A_2 A_3}^B(x_1, x_2, x_3) \\ &= \mathcal{D}^B \sum_{[C_1] \leq D_1} \sum_{[C_2] \leq [B]} \mathcal{C}_{A_1 A_2}^{C_1}(x_1, x_2) \mathcal{C}_{C_1 A_3}^{C_2}(x_2, x_3) L^{0, \Lambda_0}(\mathcal{O}_{C_2}(x_3)) \\ &+ \sum_{[C_1] \leq D_1} \mathcal{C}_{A_1 A_2}^{C_1}(x_1, x_2) \mathcal{C}_{C_1 A_3}^B(x_2, x_3) + \mathcal{D}^B(\text{partial remainder}) \end{aligned} \quad (6.158)$$

where we used eq.(6.155) in order to get rid of the other two remainder terms. Now we take  $D_1 \rightarrow \infty$  in this equation. Assuming that our spacetime arguments satisfy the condition  $\frac{|x_1 - x_2|}{|x_2 - x_3|} < 1/\tilde{K}$ , it follows from theorem 6 that the expression  $\mathcal{D}^B(\text{partial remainder})$  tends to zero.<sup>12</sup> Note that, according to theorem 6, the constant  $\tilde{K}$  depends on the loop order  $l$  and on  $n_B$ , where  $B = \{n_B, w_B\}$ . We get

$$\begin{aligned} & \mathcal{D}^B \left\{ \sum_{[C] < [B]} \mathcal{C}_{A_1 A_2 A_3}^C L^{0, \Lambda_0}(\mathcal{O}_C) \right\} + \mathcal{C}_{A_1 A_2 A_3}^B \\ &= \lim_{D_1 \rightarrow \infty} \left[ \mathcal{D}^B \left\{ \sum_{[C_1] \leq D_1} \sum_{[C_2] < [B]} \mathcal{C}_{A_1 A_2}^{C_1} \mathcal{C}_{C_1 A_3}^{C_2} L^{0, \Lambda_0}(\mathcal{O}_{C_2}) \right\} + \sum_{[C_1] \leq D_1} \mathcal{C}_{A_1 A_2}^{C_1} \mathcal{C}_{C_1 A_3}^B \right] . \end{aligned} \quad (6.159)$$

The proof can now be finished by an induction in  $[B]$ :

**Induction start** ( $[B] = 0$ ): Let  $\frac{|x_1 - x_2|}{|x_2 - x_3|} < 1/\tilde{K}$ . In this case equation (6.159) yields

$$\mathcal{C}_{A_1 A_2 A_3}^B = \lim_{D \rightarrow \infty} \sum_{[C] \leq D} \mathcal{C}_{A_1 A_2}^C \mathcal{C}_{C A_3}^B \quad (6.160)$$

as claimed in theorem 2.

**Induction step:** Assume theorem 2 holds for all  $\mathcal{C}_{A_1 A_2 A_3}^C$  with  $[C] < D_0 \in \mathbb{N}$ . Then for any  $B$  with  $[B] = D_0$  we have according to equation (6.159) (assuming also  $\frac{|x_1 - x_2|}{|x_2 - x_3|} < 1/\tilde{K}$ ,

---

<sup>12</sup>Here it is crucial that we have a bound also for arbitrary momentum derivatives of the remainder of the partial OPE, since  $\mathcal{D}^B$  also includes derivatives  $\partial_p^w$ .

which depends on  $n_B$ )

$$\begin{aligned} & \sum_{[C] < [B]} \mathcal{C}_{A_1 A_2 A_3}^C \mathcal{D}^B L^{\Lambda, \Lambda_0}(\mathcal{O}_C(0)) + \mathcal{C}_{A_1 A_2 A_3}^B \\ &= \lim_{D_1 \rightarrow \infty} \sum_{[C_1] \leq D_1} \mathcal{C}_{A_1 A_2}^{C_1} \left( \sum_{[C] < [B]} \mathcal{C}_{C_1 A_3}^C \mathcal{D}^B L^{\Lambda, \Lambda_0}(\mathcal{O}_C(0)) + \mathcal{C}_{C_1 A_3}^B \right) \end{aligned} \quad (6.161)$$

Recall, however, that our induction hypothesis  $\mathcal{C}_{A_1 A_2 A_3}^C = \lim_{D_1 \rightarrow \infty} \sum_{[C_1] \leq D_1} \mathcal{C}_{A_1 A_2}^{C_1} \mathcal{C}_{C_1 A_3}^C$  holds for all  $C$  with  $[C] < [B] = D_0$  and for  $\frac{|x_1 - x_2|}{|x_2 - x_3|} < 1/\tilde{K}_C$ , where  $\tilde{K}_C$  depends on  $C$ . Assuming  $\frac{|x_1 - x_2|}{|x_2 - x_3|} < \min_{[C] < [B]} (1/\tilde{K}_C)$ , we find that the sums over  $C$  on both sides of equation (6.161) cancel, and we are left with the claim of theorem 2.  $\square$

## 7 Summary and outlook

We have first established convergence of the OPE in Euclidean  $\varphi^4$ -theory for the expansion of  $N = 3$  fields, generalizing the statement for  $N = 2$  proved in [4]. Obviously one would like to further generalize this result to an arbitrary number  $N$  of fields, which would yield strong support to the viewpoint put forward in [12, 21, 22] that quantum field theory can be defined in terms of OPE coefficients and one-point functions as fundamental objects. This generalization appears to be possible, following a similar argument as in the present paper. However, it also requires substantial amounts of additional bookkeeping on top of the already quite heavy derivation presented here. Therefore, we will leave this topic to a future publication.

Concerning our second main result, the factorization identity satisfied by the three-point OPE coefficients, some possible lines of future work come to mind immediately. First, one would again like to generalize this result to the factorizations of an  $N$ -point coefficient. Further, it would be interesting to try and improve our bounds in order to prove factorization of the OPE not only for configurations  $|x_1 - x_2| \ll |x_2 - x_3|$ , as in theorem 2, but for any  $|x_1 - x_2| < |x_2 - x_3|$ . This would yield the stronger associativity/consistency conditions proposed in [12]. However, this task seems significantly more challenging, and it is certainly possible that one can not improve on the factorization property presented here.

**Acknowledgements:** This project was supported through ERC grant QC & C 259562. J.H. is also supported by the Leverhulme Trust grant F/00407/BM. We are grateful to Ch. Kopper for comments and discussions.



## A Proof of lemma 4

The proof follows the same strategy as the proof of lemma 4.1 in [4]. Let us assume  $x_N = 0$  for the moment. To begin with, consider the telescopic sum

$$G^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) = G_D^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) + \sum_{j=0}^D [G_{j-1}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) - G_j^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i})] \quad (\text{A.162})$$

with  $D < D' = \sum_i [A_i]$ . Further, note that for any  $j \in \mathbb{N}$  we have

$$G_{j-1}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) - G_j^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) = \sum_{C: [C]=j} \mathcal{D}^C \{F_{j-1}^{0, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i})\} L^{\Lambda, \Lambda_0}(\mathcal{O}_C(0)), \quad (\text{A.163})$$

which can be seen by checking that both sides of the equation satisfy the same linear homogeneous FE and the same boundary conditions, which are

$$\begin{aligned} \partial_{\vec{p}}^w \left( \mathcal{G}_{n, l, j-1}^{0, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}; \vec{0}) - \mathcal{G}_{n, l, j}^{0, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}; \vec{0}) \right) &= 0 & \text{for } n + |w| < j \\ \partial_{\vec{p}}^w \left( \mathcal{G}_{n, l, j-1}^{0, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}; \vec{0}) - \mathcal{G}_{n, l, j}^{0, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}; \vec{0}) \right) &= \partial_{\vec{p}}^w \mathcal{F}_{n, l, j-1}^{0, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}; \vec{0}) & \text{for } n + |w| = j \\ \partial_{\vec{p}}^w \left( \mathcal{G}_{n, l, j-1}^{\Lambda_0, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}; \vec{p}) - \mathcal{G}_{n, l, j}^{\Lambda_0, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}; \vec{p}) \right) &= 0 & \text{for } n + |w| > j \end{aligned}$$

in both cases. Further, we will need the identity

$$\begin{aligned} &\sum_{j=\Delta+1} \mathbb{T}_{\vec{x} \rightarrow \vec{0}}^j G_{D+1}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) \\ &= (-1)^{N+1} \sum_{[C]=D+1} \mathcal{D}^C \left\{ \sum_{j=\Delta+1} \mathbb{T}_{\vec{x} \rightarrow \vec{0}}^j \prod_{i=1}^N L^{0, \Lambda_0}(\mathcal{O}_{A_i}) \right\} L^{\Lambda, \Lambda_0}(\mathcal{O}_C(0)) \\ &= (-1)^{N+1} \sum_{[C]=D+1} \mathcal{D}^C \left\{ \left( 1 - \sum_{j \leq \Delta} \mathbb{T}_{\vec{x} \rightarrow \vec{0}}^j \right) \prod_{i=1}^N L^{0, \Lambda_0}(\mathcal{O}_{A_i}) \right\} L^{\Lambda, \Lambda_0}(\mathcal{O}_C(0)), \end{aligned} \quad (\text{A.164})$$

where  $\Delta = D - D'$ . The first equality follows again directly by comparison of the FE and boundary conditions of both sides of the equation. In the last line we applied the formula for the Taylor expansion with remainder, eq.(4.137). The Taylor expansion terms of degree  $j > \Delta + 1$  vanish due to the boundary conditions of the CAG's with one insertion.

Note also that the boundary conditions for the CAG's with one insertion, eq.(3.37), imply

$$\mathcal{D}^C \{G_{[C]-1}^{0, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i})\} = \mathcal{D}^C \{F_{[C]-1}^{0, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i})\} \quad \text{for } D < D'. \quad (\text{A.165})$$

We now prove lemma 4 by induction in  $D$ :

1. Induction start: For  $D = -1$  the sum in eq.(5.141) vanishes and we obtain the lemma for  $D = -1$ ,  $R_{D=-1}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) = G_{D=-1}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i})$ , trivially.
2. Induction step: Assume the lemma holds up to order  $D$ , i.e. assume

$$R_{\tilde{D}}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) = (1 - \sum_{j \leq \tilde{D} - D'} \mathbb{T}_{\vec{x} \rightarrow \vec{0}}^j) G_{\tilde{D}}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) \quad (\text{A.166})$$

for all  $\tilde{D} \leq D$ . Using again eq.(5.141), we then get

$$\begin{aligned} R_{D+1}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) &= R_D^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) - \sum_{[C]=D+1} \mathcal{C}_{A_1 \dots A_N}^C L^{\Lambda, \Lambda_0}(\mathcal{O}_C) \\ &= (1 - \sum_{j \leq \Delta} \mathbb{T}_{\vec{x} \rightarrow \vec{0}}^j) G_D^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) - \sum_{[C]=D+1} \mathcal{C}_{A_1 \dots A_N}^C L^{\Lambda, \Lambda_0}(\mathcal{O}_C) \\ &= (1 - \sum_{j \leq \Delta+1} \mathbb{T}_{\vec{x} \rightarrow \vec{0}}^j) G_{D+1}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) \\ &\quad + (1 - \sum_{j \leq \Delta} \mathbb{T}_{\vec{x} \rightarrow \vec{0}}^j) \left\{ G_D^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) - G_{D+1}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) \right\} \\ &\quad + \sum_{j=\Delta+1} \mathbb{T}_{\vec{x} \rightarrow \vec{0}}^j G_{D+1}^{\Lambda, \Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) - \sum_{[C]=D+1} \mathcal{C}_{A_1 \dots A_N}^C L^{\Lambda, \Lambda_0}(\mathcal{O}_C) \end{aligned} \quad (\text{A.167})$$

where  $\Delta = D - D'$ . Using eqs.(A.163) and (A.164) to replace the corresponding terms in the last two lines and also recalling the definition of the OPE coefficients  $\mathcal{C}_{A_1 \dots A_N}^C$ , eq.(3.62), we find that the last three terms cancel out (in the case  $\Delta < 0$  one also has to take into account eq.(A.165) to see this), leaving the claim of the lemma at order  $D + 1$  in the case  $x_N = 0$ .

The case  $x_N \neq 0$  then follows by translation covariance.  $\square$

## B Proof of lemma 5

We will divide the problem into two parts. Consider first the following contribution to the r.h.s. of eq.(6.147):

$$\begin{aligned} &[L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_2}) - L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2})] L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_3}) - \sum_{[C] \leq D} \mathcal{C}_{A_1 A_2}^C L^{\Lambda, \Lambda_0}(\mathcal{O}_C) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_3}) \\ &= -R_D^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_3}) \\ &= -(1 - \sum_{j=0}^{D-[A_1]-[A_2]} \mathbb{T}_{(x_1, x_2, x_3) \rightarrow (x_2, x_2, x_3)}^j) \left[ L_D^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}) - L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_2}) \right] L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_3}) \end{aligned} \quad (\text{B.168})$$

Here we have used lemma 4 in the last step. Now consider

$$L^{\Lambda, \Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}) - L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_3}) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_2}) - L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_2} \otimes \mathcal{O}_{A_3}) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}) \quad (\text{B.169})$$

The FE for this expression is of the form

$$\begin{aligned} \partial_\Lambda [\text{eq. (B.169)}] &= \frac{1}{2} \left\langle \frac{\delta}{\delta \varphi}, \dot{C} \star \frac{\delta}{\delta \varphi} \right\rangle [\text{eq. (B.169)}] - \left\langle \frac{\delta}{\delta \varphi} [\text{eq. (B.169)}], \dot{C} \star \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0} \right\rangle \\ &\quad - \left\langle \frac{\delta}{\delta \varphi} G^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}), \dot{C} \star \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_3}) \right\rangle, \end{aligned} \quad (\text{B.170})$$

and the boundary conditions are trivial

$$\partial_{\vec{p}}^w [\text{eq. (B.169)}]_{n,l}^{\Lambda_0, \Lambda_0}(\vec{p}) = 0 \quad \text{for all } n, l, w. \quad (\text{B.171})$$

Note that  $\sum_{[C] \leq D} \mathcal{C}_{A_1 A_2}^C L^{\Lambda, \Lambda_0}(\mathcal{O}_C \otimes \mathcal{O}_{A_3})$  obeys the same boundary conditions. Subtracting this expression from eq.(B.169) and taking the derivative with respect to  $\Lambda$  then yields

$$\begin{aligned} \partial_\Lambda [\bullet] &= \frac{1}{2} \left\langle \frac{\delta}{\delta \varphi}, \dot{C} \star \frac{\delta}{\delta \varphi} \right\rangle [\bullet] - \left\langle \frac{\delta}{\delta \varphi} [\bullet], \dot{C} \star \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0} \right\rangle \\ &\quad - \left\langle \frac{\delta}{\delta \varphi} \left[ G^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}) - \sum_{[C] \leq D} \mathcal{C}_{A_1 A_2}^C L^{\Lambda, \Lambda_0}(\mathcal{O}_C) \right], \dot{C} \star \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_3}) \right\rangle \\ &= \frac{1}{2} \left\langle \frac{\delta}{\delta \varphi}, \dot{C} \star \frac{\delta}{\delta \varphi} \right\rangle [\bullet] - \left\langle \frac{\delta}{\delta \varphi} [\bullet], \dot{C} \star \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0} \right\rangle \\ &\quad - (1 - \sum_{j=0}^{D-[A_1]-[A_2]} \mathbb{T}_{(x_1, x_2, x_3) \rightarrow (x_2, x_2, x_3)}^j) \\ &\quad \left\langle \frac{\delta}{\delta \varphi} \left[ L_D^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_2}) - L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_2}) \right], \dot{C} \star \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_3}) \right\rangle \end{aligned} \quad (\text{B.172})$$

where  $[\bullet]$  stands for the difference of eq.(B.169) and  $\sum_{[C] \leq D} \mathcal{C}_{A_1 A_2}^C L^{\Lambda, \Lambda_0}(\mathcal{O}_C \otimes \mathcal{O}_{A_3})$  and where again we used lemma 4 to simplify the remainder of the 2-point OPE. But this is exactly the FE satisfied by the expression

$$\begin{aligned} (1 - \sum_{j=0}^{D-[A_1]-[A_2]} \mathbb{T}_{(x_1, x_2, x_3) \rightarrow (x_2, x_2, x_3)}^j) &\left[ L_{\mathbf{D}}^{\Lambda, \Lambda_0}(\otimes_{i=1}^3 \mathcal{O}_{A_i}) - L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1} \otimes \mathcal{O}_{A_3}) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_2}) \right. \\ &\quad \left. - L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_2} \otimes \mathcal{O}_{A_3}) L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}) \right] \end{aligned} \quad (\text{B.173})$$

with  $\mathbf{D} = (D_{\{1,2,3\}} = -1, D_{\{2,3\}} = -1, D_{\{1,3\}} = -1, D_{\{1,2\}} = D)$ . The terms in eq.(B.173) also satisfy the trivial boundary conditions of the form (B.171). Thus, we conclude that eq.(B.169) minus  $\sum_{[C] \leq D} \mathcal{C}_{A_1 A_2}^C L^{\Lambda, \Lambda_0}(\mathcal{O}_C \otimes \mathcal{O}_{A_3})$  is equal to eq.(B.173). Putting this together with eq.(B.168) we have the proof of the lemma.  $\square$

## References

- [1] K. G. Wilson, “Renormalization group and critical phenomena. 1. Renormalization group and the Kadanoff scaling picture,” *Phys.Rev.* **B4** (1971) 3174–3183.
- [2] K. G. Wilson, “Renormalization group and critical phenomena. 2. Phase space cell analysis of critical behavior,” *Phys.Rev.* **B4** (1971) 3184–3205.
- [3] W. Zimmermann, “Lectures on elementary particles and quantum field theory, 1970 brandeis summer institute in theoretical physics,”.
- [4] S. Hollands and C. Kopper, “The operator product expansion converges in perturbative field theory,” *Commun.Math.Phys.* **313** (2012) 257–290, [arXiv:1105.3375 \[hep-th\]](#).
- [5] H. Bostelmann, “Operator product expansions as a consequence of phase space properties,” *J.Math.Phys.* **46** (2005) 082304, [arXiv:math-ph/0502004 \[math-ph\]](#).
- [6] Y.-Z. Huang and L. Kong, “Full field algebras,” *Commun.Math.Phys.* **272** (2007) 345–396, [arXiv:math/0511328 \[math.QA\]](#).
- [7] R. Guida and C. Kopper, “All-order uniform momentum bounds for the massless  $\phi^4$  theory in four dimensional Euclidean space,” [arXiv:1103.5692 \[math-ph\]](#).
- [8] J. Polchinski, “Renormalization and Effective Lagrangians,” *Nucl.Phys.* **B231** (1984) 269–295.
- [9] F. J. Wegner and A. Houghton, “Renormalization group equation for critical phenomena,” *Phys.Rev.* **A8** (1973) 401–412.
- [10] C. Wetterich, “Exact evolution equation for the effective potential,” *Phys.Lett.* **B301** (1993) 90–94.
- [11] V. F. Müller, “Perturbative renormalization by flow equations,” *Rev.Math.Phys.* **15** (2003) 491, [arXiv:hep-th/0208211 \[hep-th\]](#).
- [12] S. Hollands, “Quantum field theory in terms of consistency conditions. I. General framework, and perturbation theory via Hochschild cohomology,” [arXiv:0802.2198 \[hep-th\]](#).
- [13] R. Haag, *Local quantum physics*, 2nd edition, Springer, 1996.
- [14] G. Keller, C. Kopper, and M. Salmhofer, “Perturbative renormalization and effective Lagrangians in  $\phi^4$  in four-dimensions,” *Helv.Phys.Acta* **65** (1992) 32–52.
- [15] C. Kopper, *Renormierungstheorie mit Flussgleichungen*. Shaker, 1998.

- [16] C. Kopper, V. F. Müller, and T. Reisz, “Temperature independent renormalization of finite temperature field theory,” *Annales Henri Poincare* **2** (2001) 387–402, [arXiv:hep-th/0003254 \[hep-th\]](#).
- [17] G. Keller and C. Kopper, “Perturbative renormalization of composite operators via flow equations. 2. Short distance expansion,” *Commun.Math.Phys.* **153** (1993) 245–276.
- [18] G. Keller and C. Kopper, “Perturbative renormalization of composite operators via flow equations. 1.,” *Commun.Math.Phys.* **148** (1992) 445–468.
- [19] C. Kopper, “On the local Borel transform of Perturbation Theory,” *Commun.Math.Phys.* **295** (2010) 669–699.
- [20] G. Sansone, *Orthogonal functions*. Dover Pubns, 2004.
- [21] S. Hollands and H. Olbermann, “Perturbative Quantum Field Theory via Vertex Algebras,” *J.Math.Phys.* **50** (2009) 112304, [arXiv:0906.5313 \[math-ph\]](#).
- [22] S. Hollands and R. M. Wald, “Axiomatic quantum field theory in curved spacetime,” *Commun.Math.Phys.* **293** (2010) 85–125, [arXiv:0803.2003 \[gr-qc\]](#).